

Time-varying disaster risk models: An empirical assessment of the Rietz-Barro hypothesis*

Alfonso Irarrazabal Juan Carlos Parra-Alvarez[†]
BI Norwegian Business School Aarhus University and CREATES

February 2, 2015

Abstract

This paper revisits the fit of disaster risk models where a representative agent has recursive preferences and the probability of a macroeconomic disaster changes over time. We calibrate the model as in Wachter (2013) and perform two sets of tests to assess the empirical performance of the model in long run simulations. The model is solved using a two step projection-based method that allows us to find the equilibrium consumption-wealth ratio and dividend-yield for different values of the intertemporal elasticity of substitution. By fixing the elasticity of substitution to one, the first experiment indicates that the overall fit of the model is adequate. However, we find that the amount of aggregate stock market volatility that the model can generate is sensible to the method used to solve the model. We also find that the model generates near unit root interest rates and a puzzling ranking of volatilities between the risk free rate and the expected return on government bills. We later solve the model for values of the elasticity of substitution that differ from one. This second experiment shows that while a higher elasticity of substitution helps to increase the aggregate stock market volatility and hence to reduce the Sharpe Ratio, a lower elasticity of substitution generates a more reasonable level for the equity risk premium and for the volatility of the government bond returns without compromising the ability of the price-dividend ratio to predict excess returns.

JEL Classification: D51, E44, G12.

Keywords: Rare events, disaster risk, recursive preferences, intertemporal elasticity of substitution, projection methods, asset pricing.

*We would like to thank Juan Pablo Nicolini and participants of the DGPE workshop 2014, CREATES lunch seminar and Norges Bank internal seminar for their valuable comments. We are indebted to Jessica Wachter for sharing her codes with us. The second author acknowledges support from CREATES - Center for Research in Econometric Analysis of Time Series (DNRF78), funded by the Danish National Research Foundation, and thanks the Research Unit of Norges Bank for its hospitality during the internship program. The views expressed in this paper do not necessarily reflect the views of Norges Bank.

[†]Corresponding author. Address: Department of Economics and Business, Fuglesangs Allé 4, 8210 Aarhus V, Denmark. E-mail: jparra@econ.au.dk.

1 Introduction

Consumption capital asset pricing models (C-CAPM) constitute a fundamental tool for the study of aggregate asset prices in financial economics. They impose economic structure into the well-known capital asset pricing model (CAPM) of Sharpe (1964), Lintner (1965) and Mossin (1966) by explicitly modeling the consumption and asset choice decisions of a representative agent. Different asset pricing theories refer then to different assumptions on the shocks that drive aggregate consumption and on the preferences of the agent. In endowment economies of the Lucas (1978) type, the evolution of aggregate consumption is taken as given by the agents so when an unexpected shock hits the economy, all the adjustment to restore the general equilibrium of the economy is done through the price system.

The Rietz-Barro hypothesis, also referred to as the disaster risk model, is one of those theories. It is based on the assumption that the probability of severe economic downturns is the key to understanding the observed behavior of asset prices. The idea was originally proposed in Rietz (1988) as a response to the equity premium puzzle of Mehra and Prescott (1985). Rietz argued that the high levels of equity premium in the U.S. economy, i.e. the excess return of stocks relative to the returns offered by Treasuries, could be reconciled by models where risk averse agents demand extra compensation for the unlike, but catastrophic, events that might happen not to occur. Rietz' hypothesis was initially overlooked due to the lack of empirical evidence on the severity and duration of such disasters (See Mehra and Prescott, 1988). However, the work of Barro (2006) raised a renewed interest in rare macroeconomic disasters to explain fluctuation in aggregate asset prices. By collecting annual data on aggregate GDP growth and asset prices for twenty countries over the 20th century, he showed that the world economy has indeed experienced mayor and rather frequent depressions capable of explaining not only the high equity risk premium but also the behavior of the risk-free rate. Barro's initial work led to a fast growing literature on disaster risk models that focus on a number of extensions associated with the stochastic properties of the size and the probability of an economic catastrophe.

The theoretical framework developed in Rietz (1988) and Barro (2006) assumed a constant probability of a disaster over time. Recently, Gabaix (2012) acknowledged that this probability changes over time. Allowing for a time-varying disaster probability increases the ability of the model to match the observed behavior of asset prices in many dimensions. In fact, it has been shown that it provides a mechanism to understand what otherwise is defined as an asset pricing puzzle. In particular, the model provides a consistent framework to solve the following puzzles: (i) the equity premium puzzle of Mehra and Prescott (1985); (ii) the risk-free rate puzzle of Weil (1989); (iii) the excess volatility puzzle of Shiller (1981); (iv) the aggregate market return predictability puzzle of Campbell and Shiller (1988); (v) the cross-section predictability of stocks puzzle of Daniel and Titman (1997); (vi) the yield curve slope puzzle of Campbell (2003); (vii) the long term bond return predictability puzzle of Macaulay (1938), Fama and Bliss (1987) and Campbell and Shiller (1991); (viii) the credit spread puzzle of Almeida and Philippon (2007); (ix) the deep out-of-the money put prices vs. Black-Scholes mismatch of Jackwerth and Rubinstein (1996); and (x) the positive relation between high put option prices and their high future returns of Bollerslev et al. (2009).

The time-varying disaster risk model has been also studied in Wachter (2013). She shows that this

model can explain many features of the aggregate stock market and in particular it is able to generate high level of stock market volatility. Her model also accounts for the predictability of excess returns by valuation ratios without generating counterfactual long-run predictability in consumption and dividend growth. To do so, she employs an endowment economy in which aggregate consumption growth is subject to discrete, permanent and instantaneous downward jumps with time-varying arrival rate, and where the representative investor has recursive preferences. Her setup allows for an approximated analytical solution when the intertemporal elasticity of substitution is equal to one.

The purpose of this paper is twofold. We first revisit the empirical fit of the time-varying disaster risk model using the framework and calibration of Wachter (2013). We depart from her setup in the way the general equilibrium of the model is approximated. Instead of using her approximated analytical solution, we use a two step projection method to solve for the consumption-wealth ratio and the price-dividend ratio that ensure no arbitrage in each of the asset classes. The method used in this paper provides arbitrary accuracy and low computation time for any set of model parameters including the case of intertemporal elasticity of substitution above or below unity. Projection methods have been recently studied in discrete-time models with recursive preferences by Pohl et al. (2014) for endowment asset pricing models and in Caldara et al. (2012) and van Binsbergen et al. (2012) for DSGE models.

Second, we use the proposed solution method to solve the model for different values of the intertemporal elasticity of substitution (IES) while keeping the remaining parameters as in Wachter (2013). We then study the implications of different IES for asset pricing within the time-varying disaster risk framework. While unitary IES is a convenient assumption to obtain closed form expressions for asset prices, the evidence on its correct value is mixed and its choice is usually made by fixing its value such that some features of the data can be replicated. In the asset pricing literature, and more recently in the macro-financial literature, it is customary to impose values for the IES greater than one in order to generate predictability of excess returns and countercyclicality of the equity risk premium. However, when using aggregate macroeconomic data, the IES is usually estimated to be below one or close to zero as shown in Hall (1988). More recently, Havranek et al. (2014) found an average estimate of 0.5 in a comprehensive meta-analysis of more than 2500 available estimates from 169 published studies¹.

Our results confirm the virtues of the time-varying disaster risk and reveals some of its limitations. We compare simulations of the model with those from the data conditional on no crisis. For the case of unitary IES, we find that the solution method is crucial to generate aggregate stock market volatility close to that observed in the data. Whereas the approximated closed form solution of Wachter (2013) generates a standard deviation of stock returns of around 17.66% per year, the two step projection method only produces a annual volatility of 11.54%. While still being an outstanding result relative to other asset pricing theories, we stress the importance of not overstating the advantages of the time-varying disaster risk model. We also find that the model generates near unit root risk free and government bill returns which is at odds with the U.S. postwar data. The model also generates the counterfactual implication that the volatility of the risk free rate is higher than that on the returns of Treasury bills.

Our numerical experiments show that opposite to previous results in the literature, the predictability

¹Half of the values used in their analysis correspond to estimates for the U.S. economy.

properties of the time-varying disaster risk model do not require the IES to be greater or equal to one. In fact, for the cases studied in the paper we always find a negative correlation between the price-dividend ratio and excess returns. These results suggest that a low IES it is consistent with a countercyclical equity risk premium as long as the representative agent prefers an early resolution of uncertainty. In general, we find that an IES lower than one has the potential effect of generating appropriate levels of equity risk premium and volatility of government bonds, while an IES greater than one can generate higher price-dividend volatility and stock market volatility, and hence a lower Sharpe Ratio.

The remainder of the paper is organized as follows. In section (2) we present the model and derive general equilibrium results for various assets in the economy. In Section (3) we present the general solution algorithm. Section (4) discusses the calibration of the model and investigates its the empirical implications. Finally Section (5) concludes.

2 Model

We study a continuous-time representative agent, fruit-tree model of asset pricing with exogenous, stochastic endowment. Our economy coincides with that studied in Wachter (2013). However, we do not require the IES to be equal to one. The endowment of fruit in any period t is given by Y_t , and the number of trees is fixed. Since we assume that the economy is closed and there is no investment, neither depreciation, all output is consumed, that is, $C_t = Y_t$ at each point in time.

2.1 Endowment process

In what follows we assume that the growth rate of aggregate consumption in the economy is subject to both continuous fluctuations and discrete, instantaneous and permanent downwards jumps. The latter are modeled through a Poisson process, N_t , with time-varying arrival rate λ_t , that captures the idea of low-probability disasters. Large downward jumps are regarded as disasters or catastrophic events. Moreover, the Poisson Zero-One Jump Law guarantees that only one disaster can occur per unit of time (See Hanson, 2007). If a disaster occurs, consumption falls by a fraction $(1 - Z_t)$, where $0 < Z_t \leq 1$ is a random variable with well-defined probability density function $\zeta(Z_t)$. On the other hand, the continuous fluctuations are captured by a standard Brownian motion, B_t . Therefore, the dynamics of aggregate consumption, C_t , can be represented by the jump-diffusion process:

$$dC_t = \mu C_{t-} dt + \sigma C_{t-} dB_t - (1 - Z_t) C_{t-} dN_t \quad (2.1)$$

where C_{t-} denotes the level of consumption before a jump occurs, and μ and σ capture the constant mean and volatility of consumption growth in normal times, i.e. when no disaster occurs². The assumptions

²An alternative way to define an economic disaster would be to introduce a small probability of declines in the average growth rate of consumption, μ . This would accommodate the definition of great depression used in Kehoe and Prescott (2002): “... *First, it must be a sufficiently large deviation. Our working definition is that a great depression is a deviation at least 20% below trend. Second, the deviation must occur rapidly. Our working definition is that detrended output per working-age person must fall at least 15% within a decade of the depression.*”

in Equation (2.1) differ from those in the long-run risk model of Bansal and Yaron (2004) where it is assumed that both mean and volatility of consumption growth change over time while no role is given to rare events.

We assume that the Poisson arrival rate or disaster probability, λ_t , follows a square-root process of the form:

$$d\lambda_t = \kappa (\bar{\lambda} - \lambda_t) dt + \sigma_\lambda \sqrt{\lambda_t} dB_{\lambda,t} \quad (2.2)$$

where $B_{\lambda,t}$ is a standard Brownian motion. Equation (2.2) is a continuous-time version of an heteroskedastic autoregressive process with mean reversion parameter κ , unconditional mean $\bar{\lambda}$, and scale parameter σ_λ . The square-root process ensures that the random variable λ_t only takes positive values. In fact, if $\lambda_t \rightarrow 0$, then the drift term $\kappa \bar{\lambda} > 0$ and the variance $\sigma_\lambda^2 \lambda_t \rightarrow 0$, implying that the process will remain positive w.p.1. Furthermore, if we add the restriction $\sigma_\lambda^2 < 2\kappa \bar{\lambda}$, then the process never reaches zero: in this case, the drift will be very high relative to the variance, shooting up the process even before it reaches zero. In addition to this, the positive skewness, excess kurtosis and the boundedness from below of the λ_t -process implies large but infrequent declines or disasters in consumption. Finally and for simplicity, we assume independence between B_t , $B_{\lambda,t}$, N_t and Z_t .

2.2 Preferences

The representative agent of this economy has preferences over streams of consumption described by the stochastic differential utility framework developed in Duffie and Epstein (1992a,b). These preferences correspond to the continuous-time version of the recursive preferences proposed in Kreps and Porteus (1978), Epstein and Zin (1989) and Weil (1989) which generalize the standard case of power utility by disentangling the agent's risk aversion from the intertemporal elasticity of substitution.

Let V_t denote the utility function of the agent at time t . Then, his preferences are described by:

$$V_t = \mathbb{E}_t \left[\int_t^\infty f(C_s, V_s) ds \right] \quad (2.3)$$

where $f(C_s, V_s)$ is a normalized aggregator of current consumption, C_s , and continuation utility, V_s , for $s \geq t$. In its more general form, the aggregator is given by:

$$f(C_s, V_s) = \beta \theta V_s \left[\left(\frac{C_s}{((1-\gamma) V_s)^{\frac{1}{1-\gamma}}} \right)^{1-\frac{1}{\psi}} - 1 \right]. \quad (2.4)$$

Here $\beta > 0$ is the rate of time preference, $\gamma > 0$ denotes the agent's coefficient of relative risk aversion, and $\psi > 0$ his intertemporal elasticity of substitution. We have also defined the constant $\theta = \frac{1-\gamma}{1-\psi^{-1}}$. If $\gamma > \psi^{-1}$, the agent prefers early over later resolution of uncertainty. Furthermore, for the relevant case of $\gamma > 1$, the parameter $\theta < 0$ if and only if $\psi > 1$. In the limiting case of unitary IES the aggregator reduces to:

$$f(C_s, V_s) = \beta(1 - \gamma) V_s \left(\log C_s - \frac{1}{1 - \gamma} \log((1 - \gamma) V_s) \right). \quad (2.5)$$

The standard case of power time-additive separable preferences is nested in Equation (2.4) and can be obtained whenever $\psi = 1/\gamma$. In this case, the agent is indifferent about the timing of the resolution of uncertainty. The properties of the stochastic differential utility in Equation (2.3) for the case of jump-diffusion process like the one specified in Equation (2.1) have been previously studied in Benzoni et al. (2011).

2.3 Model implications

The model outlined previously can be solved as a social planning problem in which the competitive equilibrium is reached by allocating all the endowment to consumption. Solving for the competitive equilibrium amounts to finding economic quantities such as the consumption-wealth ratio and/or the price-dividend ratio that ensure complete markets and no-arbitrage opportunities among the different asset classes. Once we have solved for the equilibrium allocations, we can use the representative agent's marginal utility to derive the equilibrium stochastic discount factor (SDF) that prices all the assets in the economy. In particular, we will be interested in pricing claims to aggregate consumption, claims to aggregated dividends, a riskless bond and a partially defaultable short-term government bond. Under recursive preferences, the agent's marginal utility depends both on aggregate consumption and continuation utility. Therefore, to compute the equilibrium asset prices we first need to find the optimal value function which measures the utility obtained by the representative agent for any given path of the aggregate consumption in Equation (2.1).

2.3.1 Optimal value function

In general equilibrium the optimal allocation of the endowment satisfies the Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = f(C_t, V(C_t, \lambda_t; \psi)) + \mu C_t V_C(C_t, \lambda_t; \psi) + \kappa(\bar{\lambda} - \lambda_t) V_\lambda(C_t, \lambda_t; \psi) + \frac{1}{2} C_t^2 \sigma^2 V_{CC}(C_t, \lambda_t; \psi) + \frac{1}{2} \sigma_\lambda^2 \lambda_t V_{\lambda\lambda}(C_t, \lambda_t; \psi) + \lambda_t \mathbb{E}_\zeta(V(Z_t C_t, \lambda_t; \psi) - V(C_t, \lambda_t; \psi)) \quad (2.6)$$

where $\mathbb{E}_\zeta(\cdot)$ is the expectation operator taken with respect to the distribution function $\zeta(Z_t)$, and where we have used the fact that the utility function in Equation (2.3) can be written as $V(C_t, \lambda_t; \psi)$ given the Markov properties of C_t and λ_t . The dependence of the value function on the IES is made explicit for interpretation purposes in later sections of the paper where we consider the implications of the model for different values of this parameter. As shown in Appendix (A.1), the utility function is homogeneous of degree $(1 - \gamma)$ in consumption which implies that Equation (2.6) can be reduced to a quasi-linear differential equation in λ_t . In particular, by choosing:

$$V(C_t, \lambda_t; \psi) = \frac{1}{1-\gamma} C_t^{1-\gamma} H(\lambda_t; \psi), \quad (C, \lambda) \in (0, \infty) \times \mathbb{R}_+ \quad (2.7)$$

the HJB equation becomes:

$$0 = \frac{f(C_t, V(C_t, \lambda_t; \psi))}{V(C_t, \lambda_t; \psi)} H(\lambda_t; \psi) + \left[(1-\gamma)\mu - \frac{1}{2}\gamma(1-\gamma)\sigma^2 + \lambda_t \mathbb{E}_\zeta \left(Z_t^{1-\gamma} - 1 \right) \right] H(\lambda_t; \psi) \\ + \kappa(\bar{\lambda} - \lambda_t) H_\lambda(\lambda_t; \psi) + \frac{1}{2} \sigma_\lambda^2 \lambda_t H_{\lambda\lambda}(\lambda_t; \psi) \quad (2.8)$$

where the non-linear term is given by

$$\frac{f(C_t, V(C_t, \lambda_t; \psi))}{V(C_t, \lambda_t; \psi)} H(\lambda_t; \psi) = \begin{cases} \frac{\beta}{1-q} H(\lambda_t; \psi)^q - \frac{\beta}{1-q} H(\lambda_t; \psi) & \text{if } \psi \neq 1 \\ -\beta H(\lambda_t; \psi) \log H(\lambda_t; \psi) & \text{if } \psi = 1 \end{cases}$$

with $q = 1 - \frac{1}{\theta}$. We will refer to Equation (2.8) as the reduced HJB equation. A complete derivation of Equation (2.8) can be found in Appendix (A.2). The function $H(\lambda_t; \psi)$ defines the elasticity of the life-time utility of the representative investor with respect to the disaster risk:

$$\frac{\partial V(C_t, \lambda_t; \psi)}{\partial \lambda_t} \frac{\lambda_t}{V(C_t, \lambda_t; \psi)} = \lambda_t \frac{H_\lambda(\lambda_t; \psi)}{H(\lambda_t; \psi)}$$

and its magnitude will ultimately depend on the level of IES.

2.3.2 Stochastic discount factor, risk free rate and partially defaultable liabilities

Standard asset pricing theory suggests that in the absence of arbitrage and complete markets, the price P_t of any given asset in the economy that pays dividends at a rate D_s for $s \geq t$ solves the present-value relation:

$$P_t = \mathbb{E}_t \left[\int_t^\infty \frac{\pi_s}{\pi_t} D_s ds \right], \quad (2.9)$$

where π_t denotes the equilibrium stochastic discount factor (SDF), or pricing kernel of the economy. Equation (2.9) can be rewritten as a partial differential equation of the form:

$$0 = \pi_t D_t + \mathbb{E}_t [d(\pi_t P_t)] \quad (2.10)$$

which becomes useful for some of the derivations below.

Duffie and Skiadas (1994) have shown that when the representative agent has recursive preferences as in Equation (2.3), the SDF is given by:

$$\pi_t(\lambda_t; \psi) = \exp \left\{ \int_0^t f_V(C_s, V(C_s, \lambda_s; \psi)) ds \right\} f_C(C_t, V(C_t, \lambda_t; \psi)) \quad (2.11)$$

where $f_V(\cdot, \cdot)$ and $f_C(\cdot, \cdot)$ are the derivatives of utility aggregator with respect to the value function and the aggregate consumption respectively. For a given value of $H(\lambda_t; \psi)$, the following proposition provides a characterization of the level and dynamics of the equilibrium SDF together with the equilibrium risk free rate.

Proposition 2.1 (Stochastic Discount Factor and Risk free rate). *Given the optimal value of $H(\lambda_t; \psi)$, the SDF for this economy is given by:*

$$\pi_t(\lambda_t; \psi) = \begin{cases} \exp \left\{ \frac{\beta}{1-q} \int_0^t [qH(\lambda_s; \psi)^{q-1} - 1] ds \right\} \beta C_t^{-\gamma} H(\lambda_t; \psi)^q, & \text{if } \psi \neq 1 \\ \exp \left\{ -\beta \int_0^t [\log H(\lambda_s; \psi) + 1] ds \right\} \beta C_t^{-\gamma} H(\lambda_t; \psi), & \text{if } \psi = 1. \end{cases} \quad (2.12)$$

The dynamics of the SDF is governed by:

$$\frac{d\pi_t(\lambda_t; \psi)}{\pi_t(\lambda_t; \psi)} = \mu_{\pi,t}(\lambda_t; \psi) dt + \boldsymbol{\sigma}_{\pi,t}(\lambda_t; \psi) \left[dB_t \quad dB_{\lambda,t} \right]' + (Z_t^{-\gamma} - 1) dN_t \quad (2.13)$$

where $\mu_{\pi,t}(\lambda_t; \psi)$ and $\boldsymbol{\sigma}_{\pi,t}(\lambda_t; \psi)$ are given by:

$$\mu_{\pi,t}(\lambda_t; \psi) = \begin{cases} -\frac{\beta}{1-q} - \gamma\mu + \frac{1}{2}\gamma(1+\gamma)\sigma^2 + q \left[\frac{\beta}{1-q} H(\lambda_t; \psi)^{q-1} \right. \\ \quad \left. + \kappa(\bar{\lambda} - \lambda_t) \frac{H_{\lambda}(\lambda_t; \psi)}{H(\lambda_t; \psi)} + \frac{1}{2} \left((q-1) \left(\frac{H_{\lambda}(\lambda_t; \psi)}{H(\lambda_t; \psi)} \right)^2 + \frac{H_{\lambda\lambda}(\lambda_t; \psi)}{H(\lambda_t; \psi)} \right) \sigma_{\lambda}^2 \lambda_t \right] & \text{if } \psi \neq 1 \\ -\beta - \gamma\mu + \frac{1}{2}\gamma(1+\gamma)\sigma^2 \\ \quad -\beta \log H(\lambda_t; \psi) + \kappa(\bar{\lambda} - \lambda_t) \frac{H_{\lambda}(\lambda_t; \psi)}{H(\lambda_t; \psi)} + \frac{1}{2} \frac{H_{\lambda\lambda}(\lambda_t; \psi)}{H(\lambda_t; \psi)} \sigma_{\lambda}^2 \lambda_t & \text{if } \psi = 1 \end{cases} \quad (2.14)$$

and

$$\boldsymbol{\sigma}_{\pi,t}(\lambda_t; \psi) = \begin{cases} \left[-\gamma\sigma \quad q \frac{H_{\lambda}(\lambda_t; \psi)}{H(\lambda_t; \psi)} \sigma_{\lambda} \sqrt{\lambda_t} \right] & \text{if } \psi \neq 1 \\ \left[-\gamma\sigma \quad \frac{H_{\lambda}(\lambda_t; \psi)}{H(\lambda_t; \psi)} \sigma_{\lambda} \sqrt{\lambda_t} \right] & \text{if } \psi = 1 \end{cases} \quad (2.15)$$

respectively.

Consider now the case of an asset that has a constant price equal to one and pays the risk free rate, r_t^f , as a dividend. Then, Equations (2.9) and (2.10) imply that:

$$r_t^f(\lambda_t; \psi) = -\mu_{\pi,t}(\lambda_t; \psi) - \lambda_t \mathbb{E}_{\zeta} (Z_t^{-\gamma} - 1). \quad (2.16)$$

Proof. See Appendix (A.3) □

As documented in Barro (2006), economic disasters are usually followed by partial default or loss of property rights on the risk free investment. One way in which the government defaults on its debt is through the depreciation of its real value via high inflation. Hence, allowing for some form of partial default is of empirical relevance in order to match the model's risk free asset to the observed rate of return on government securities in the data.

Let $L_t(\lambda_t; \psi)$ be the price process resulting from rolling over instantaneous government debt:

$$\frac{dL_t(\lambda_t; \psi)}{L_t(\lambda_t; \psi)} = r_t^L(\lambda_t; \psi) dt - (1 - Z_{L,t}) dN_t \quad (2.17)$$

where $r_t^L(\lambda_t; \psi)$ is the amount investors receive if there is no default. In the event of a disaster, there will be a partial default on government liabilities with constant probability $p \geq 0$ where the fraction $(1 - Z_{L,t})$ of the bill's gross return is lost. If the likelihood of default is zero, then the government bill pays the risk free rate in Equation (2.16). We follow Barro (2006) and Wachter (2013) and assume that in the event of default, the percentage loss is equal to the percentage decline in consumption:

$$Z_{L,t} = \begin{cases} Z_t & \text{with probability } p \\ 1 & \text{otherwise.} \end{cases}$$

Equation (2.10) states that the price of the government liability must satisfy:

$$0 = r_t^L(\lambda_t; \psi) + \mu_{\pi,t}(\lambda_t; \psi) + \lambda_t \mathbb{E}_\zeta [Z_t^{-\gamma} Z_{L,t} - 1]$$

where:

$$\mathbb{E}_\zeta [Z_t^{-\gamma} Z_{L,t} - 1] = p \mathbb{E}_\zeta [Z_t^{1-\gamma} - 1] + (1-p) \mathbb{E}_\zeta [Z_t^{-\gamma} - 1].$$

Then, the equilibrium rate promised by governments on short-term debt can be characterized by the following proposition.

Proposition 2.2 (Return on short-term government liabilities). *The instantaneous interest rate on defaultable short-term debt issued by the government is given by:*

$$r_t^L(\lambda_t; \psi) = r_t^f(\lambda_t; \psi) + \lambda_t p \mathbb{E}_\zeta [Z_t^{-\gamma} (1 - Z_t)]. \quad (2.18)$$

Let $r_t^b(\lambda_t; \psi)$ denote the expected return on government liabilities which includes an adjustment on the promised rate of return $r_t^L(\lambda_t; \psi)$ for the unconditional expected loss in the event of a government default. Hence:

$$r_t^b(\lambda_t; \psi) = r_t^f(\lambda_t; \psi) + \lambda_t p \mathbb{E}_\zeta [(Z_t^{-\gamma} - 1)(1 - Z_t)]. \quad (2.19)$$

Proof. See Appendix (A.4). □

The second term in Equation (2.19) has the interpretation of a disaster risk premium required by

investors in order to hold government bonds relative to an otherwise risk free asset: the percentage change in marginal utility is multiplied by the loss in the asset.

2.3.3 Returns on consumption claims, dividend claims and risk premiums

Let S_t denote the value of a claim to aggregate consumption. In general equilibrium, S_t equals the representative agent's wealth W_t . Hence, Equation (2.9) states that the aggregate level of wealth at time t is given by the present value of all future expected consumption. In other words, we are pricing a financial asset that pays aggregate consumption as its dividend. The valuation ratio for this asset is given by the wealth-consumption ratio and it is related to the unknown function $H(\lambda_t; \psi)$. The following proposition defines this relation and the consumption risk premium.

Proposition 2.3 (Wealth-consumption ratio and consumption risk premium). *The value of a claim to aggregate consumption is given by:*

$$W_t = \frac{1}{\beta} C_t H(\lambda_t; \psi)^{1-q}. \quad (2.20)$$

When $\psi = 1$, $1 - q = 0$ and the wealth-consumption ratio becomes:

$$W_t = \frac{1}{\beta} C_t. \quad (2.21)$$

Now let rp_t^C denote the excess return of an asset that pays consumption as dividend relative to the risk free rate. The instantaneous consumption risk premia is given by:

$$rp_t^C(\lambda_t; \psi) = \gamma\sigma^2 - q(1-q) \left(\frac{H_\lambda(\lambda_t; \psi)}{H(\lambda_t; \psi)} \right)^2 \sigma_\lambda^2 \lambda_t + \lambda_t \mathbb{E}_\zeta [(Z_t^{-\gamma} - 1)(1 - Z_t)]. \quad (2.22)$$

Equation (2.22) is inclusive of jumps. The consumption risk premium conditional on no disasters, which we denote by $\tilde{r}p_t^C(\lambda_t; \psi)$, is given by:

$$\tilde{r}p_t^C(\lambda_t; \psi) = rp_t^C(\lambda_t; \psi) + \lambda_t \mathbb{E}_\zeta [1 - Z_t]. \quad (2.23)$$

Proof. See Appendix (A.5). □

The first two terms in Equation (2.22) represent the portion of the consumption risk premium that is compensation for diffusion risk, which includes a compensation for the time-variation in λ_t . The last term is the compensation for jump risk. In general, $\tilde{r}p_t^C > rp_t^C$ given the reduction in realized returns when a disaster occurs.

While the first and last terms of Equation (2.22) contribute positively to the consumption risk premium, the compensation for fluctuations in the disaster risk depends on the sign of $q(1-q)$. Consider the case of power utility which is obtained when $\gamma = \psi^{-1}$. In this case $q(1-q) = 0$ implying that investors are not compensated for fluctuations in the probability of a disaster. For a given level of risk aversion, the same result holds under recursive preferences and unitary IES. In this scenario the substitution and wealth effects offset each other implying that the consumption-wealth ratio is constant to variations in

the disaster risk. Therefore, no extra compensation for changes in the probability of a disaster is required by the investor in order to hold the risky asset.

Now consider the case of IES different from one. The compensation for time variation in the disaster risk will depend on whether the agent prefers early or late resolution of uncertainty. It can be shown that $q(1-q) < 0$ if and only if $\gamma > \psi^{-1}$, that is, if the representative agent prefers early resolution of uncertainty. However, if $\gamma < \psi^{-1}$, then the agent will prefer late resolution of uncertainty which implies $q(1-q) > 0$, suggesting that the risk premium on consumption claims relative to the risk free will be lower.

Following Wachter (2013) we model the aggregate stock market by assuming that its cash flow at time t , D_t , is related to the level of aggregate consumption as in Abel (1999) and Campbell (2003). In particular we assume that $D_t = C_t^\varphi$, where $\varphi > 1$ is a leverage parameter. Itô's lemma implies that:

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma \varphi dB_t - (1 - Z_t^\varphi) dN_t \quad (2.24)$$

where

$$\mu_D = \varphi \mu + \frac{1}{2} \sigma^2 \varphi (\varphi - 1). \quad (2.25)$$

A simple log transformation shows that under the maintained assumption of $\varphi > 1$ dividends fall by more than consumption in the event of a disaster.

Let $F_t(\lambda_t; \psi)$ denote the price of a claim to future dividends and $G(\lambda_t; \psi)$ the log price-dividend ratio. Thus:

$$F_t(\lambda_t; \psi) = D_t \exp[G(\lambda_t; \psi)]. \quad (2.26)$$

The martingale property implicit in Equation (2.10) together with the dynamics of the SDF in Equation (2.13) and the dynamics of the equity price yield a quasi-linear ODE for the log price-dividend ratio. The following proposition defines the equilibrium price-dividend ratio and provides an expression for the equity risk-premium relative to the risk free asset, and to the expected return on government liabilities.

Proposition 2.4 (Price-dividend ratio and equity risk-premium). *For a given value of $H(\lambda_t; \psi)$, the log price-dividend ratio $G(\lambda_t; \psi)$ is the solution to the ODE:*

$$\begin{aligned} & \frac{1}{\exp(G(\lambda_t; \psi))} - r_t^f(\lambda_t; \psi) + \lambda_t \mathbb{E}_\zeta \left(Z_t^{(\varphi-\gamma)} - Z_t^{-\gamma} \right) + \varphi \mu + \frac{1}{2} \sigma^2 \varphi (\varphi - 1 - 2\gamma) \\ & + \left(\kappa (\bar{\lambda} - \lambda_t) + q \frac{H_\lambda(\lambda_t; \psi)}{H(\lambda_t; \psi)} \sigma_\lambda^2 \lambda_t \right) G_\lambda(\lambda_t; \psi) + \frac{1}{2} \sigma_\lambda^2 \lambda_t G_{\lambda\lambda}(\lambda_t; \psi) + \frac{1}{2} \sigma_\lambda^2 \lambda_t G_\lambda(\lambda_t; \psi)^2 = 0. \end{aligned} \quad (2.27)$$

The instantaneous equity risk premium with respect to the risk free rate is given by:

$$rp_t^D(\lambda_t; \psi) = \varphi \gamma \sigma^2 - q \frac{H_\lambda(\lambda_t; \psi)}{H(\lambda_t; \psi)} G_\lambda(\lambda_t; \psi) \sigma_\lambda^2 \lambda_t + \lambda_t \mathbb{E}_\zeta \left[(Z_t^{-\gamma} - 1) (1 - Z_t^\varphi) \right]. \quad (2.28)$$

Equation (2.28) is inclusive of jumps. The equity risk premium conditional on no disasters, which we denote by $\tilde{r}p_t^D(\lambda_t; \psi)$, is given by:

$$\tilde{r}p_t^D(\lambda_t; \psi) = rp_t^D(\lambda_t; \psi) + \lambda_t \mathbb{E}_\zeta [1 - Z_t^\varphi]. \quad (2.29)$$

The instantaneous equity risk premium with respect to the return on government bills is given by:

$$\begin{aligned} rp_{b,t}^D(\lambda_t; \psi) = & \varphi\gamma\sigma^2 - q \frac{H_\lambda(\lambda_t; \psi)}{H(\lambda_t; \psi)} G_\lambda(\lambda_t; \psi) \sigma_\lambda^2 \lambda_t \\ & + \lambda_t \mathbb{E}_\zeta [(Z_t^{-\gamma} - 1) ((1-p)(1-Z^\varphi) + p(Z - Z^\varphi))] \end{aligned} \quad (2.30)$$

Proof. See Appendix (A.6). □

The first two terms in Equations (2.28) and (2.30) represent that part of the equity risk premium that constitutes compensation for the risk in normal times. The third term is the compensation for jump risk. The latter is indeed a premium for catastrophic events since it is made of changes in marginal utility and the potential losses in the asset due to a disaster.

Similar to the consumption risk premium, the first and third terms in Equations (2.28) and (2.30) contribute positively to the equity risk premium. However, the compensation effect from time-variation in the probability of a disaster is not clear at a glance. As it will be shown below, its contribution is always positive as long as the agent prefers early resolution of uncertainty ($q > 0$) since this will generate, consistent with the empirical evidence, a decreasing price-dividend ratio, $G_\lambda < 0$, and an increasing $H_\lambda > 0$.

As argued in Wachter (2013), introducing stochastic variation in the disaster risk generates time variation in the valuation ratios, the risk premium and the interest rates allowing us to test the predictability properties of Rietz-Barro hypothesis. An additional feature that makes the model attractive is the fact that it creates not just time variation but also higher levels of volatility in the aggregate stock market. Using the implied dynamics of Equation (2.26), the instantaneous volatility of the stock market conditional on no jumps occurring is given by:

$$(\sigma_{F,t} \sigma'_{F,t})^{\frac{1}{2}} = \left(\varphi^2 \sigma^2 + G_\lambda(\lambda_t; \psi)^2 \sigma_\lambda^2 \lambda_t \right)^{\frac{1}{2}}. \quad (2.31)$$

Relative to a model with no disasters as in Mehra and Prescott (1985), and to those with constant probability of a disaster as in Rietz (1988) and Barro (2006), the time varying disaster risk model generates stock market volatility that exceeds that of dividends as long as $\lambda_t > 0$.

3 Model solution

As discussed in Section (2) a solution to the time-varying disaster risk model is given by the unknown functions $H(\lambda_t; \psi)$ and $G(\lambda_t; \psi)$. Once they have been computed, it is possible to approximate the

consumption-wealth ratio and the price-dividend ratio. In general, the model does not admit a closed form solution. However, Wachter (2013) shows that it is possible to derive an exact analytical solution for $H(\lambda_t; \psi)$ and an approximate closed form solution for $G(\lambda_t; \psi)$ for the case $\psi = 1$. The latter is approximate since it can only be computed in close form up to a definite integral that has to be numerically approximated. The approximation method used in Wachter (2013) was first introduced for models with external habits in Wachter (2005).

In the more general case of $\psi \neq 1$, no closed form solution is available and hence it is necessary to resort to numerical approximation methods to find the optimal allocations. The standard approach used in the asset pricing literature is to derive approximate analytical solutions using log-linear approximation techniques. This method, based on the Campbell and Shiller decomposition (see Campbell and Shiller, 1988), has been implemented in discrete time dynamic asset allocation models by Campbell and Viceira (2002) and Chacko and Viceira (2005) and by Campbell et al. (2004) for continuous time models. An application to the time-varying disaster risk model can be found in Nowotny (2011).

While linearization methods are very fast and accurate around the point of reference, the approximation errors can become fairly large for more complex models where nonlinearities play an important role. In fact, Kraft et al. (2014) use the time-varying disaster risk model of Wachter (2013) to show that, for a risk aversion of 6, the Campbell and Shiller approximation deviates considerably from the true solution for values of the IES greater than 1.4. Similar conclusions have been obtained in Pohl et al. (2014) for the models of Tallarini (2000) and Bansal and Yaron (2004). To overcome these difficulties, we propose the use of global methods which are both efficient and accurate, and robust to changes in the model specification. They also provide an arbitrary degree of accuracy in the whole state space of the unknown functions. In particular, we use collocation methods which were first introduced in economics by Judd (1992), and have been extensively used by macroeconomists and financial economists to solve fixed point problems. A general treatment of the collocation method can be found in Judd (1998) and in Miranda and Fackler (2002). Formally, we use a two-step collocation method in line with the procedure described in Pohl et al. (2014).

In a first step, we solve for the $H(\lambda_t; \psi)$ function. As seen from Equation (2.8) this function must satisfy the reduced HJB equation. Since it is not possible to solve an infinite dimensional problem numerically, we use Chebyshev basis functions to approximate the unknown function by the closest function in the polynomial space. The approximation to the $H(\lambda_t; \psi)$ function and its derivatives are then plugged back into the original functional equation and solved for exactly at a discrete set of grid points.

Once the approximation for $H(\lambda_t; \psi)$ is obtained, we proceed to compute the risk free rate using Equations (2.14) and (2.16) at each point in the grid. Using these approximants as input, the second step solves for the unknown function $G(\lambda_t; \psi)$ by using once again Chebyshev basis functions. The approximation to the $G(\lambda_t; \psi)$ function and its derivatives are then plugged back into Equation (2.27) and solved for exactly at the same discrete set of grid points used in the first step. The grid points for both steps in the algorithm are chosen as the zeros of the Chebyshev polynomials; this choice of polynomial representation and choice of grid points ensures that the numerical problem is well conditioned. A

complete description of the algorithm is given in Table (1).

[Table 1 about here.]

Figure (1) plots the annual log consumption-wealth ratio, $(c_t - w_t)$, and the annual log price-dividend ratio, $(f_t - d_t)$, for different values of the IES as a function of the disaster risk λ_t . Both ratios have been normalized by their corresponding values when the probability of a disaster is null³. Under the standing assumption that $\gamma > \psi^{-1}$, the bottom panel indicates that higher disaster risk implies lower prices of the claims to aggregated dividends. This result is independent of the value of the IES. It also indicates that the degree by which equity prices fall in response to increases in the probability of a disaster is negatively related to the IES. Contrary to the case where λ_t is assumed to be i.i.d. (see Gourio, 2008), the price-dividend ratio always decreases as the disaster risk increases.

[Figure 1 about here.]

The same does not hold for claims to aggregate consumption. The time-varying disaster risk model generates three different correlations between the disaster risk and the consumption-wealth ratio which depends on the value of the IES relative to unity. These scenarios are directly related to the way aggregate wealth adjust when the perceived probability of a disaster changes. The way in which the price to consumption claims react is ultimately related to the interplay between the substitution and income effects. If agents have an unitary IES then both effects cancel out, making the agent indifferent to changes in λ_t . However, if $\psi > 1$, the substitution effect will dominate and for a given level of consumption, prices should fall. The opposite occurs when $\psi < 1$. The intuition behind this reasoning will be further analyzed in Section (4).

The role played by the IES is better understood by looking at the implications of the model for asset prices. Figure (2) plots the expected return on government bills, the expected returns on equity, the equity risk premium and the aggregate stock market volatility for different values of the IES as a function of the disaster risk, λ_t . The graphs reveal a monotonic relation between each of the variables under consideration and the disaster intensity which is independent of the IES.

[Figure 2 about here.]

In panel (a) we see that the the expected rate of return on government bills is always decreasing in λ_t . First, by definition (as in Equation (2.19)) it corresponds to the sum of the risk free rate and an adjustment term from the probability of a default. A higher probability of disaster reduces the expected growth rate in the economy leading the representative agent to increase his savings and reducing this way the return on the risk free asset. On the other hand a high probability of default leads to the agent to demand a higher return to hold the asset.

Panel (b) shows that the expected return on equity is always increasing in λ_t . This implication is opposite to that obtained in the case of i.i.d. λ_t , where Gourio (2008) has shown that equity returns

³The final degree of the approximation for $\hat{H}(\lambda_t; \omega^H)$ and $\hat{G}(\lambda_t; \omega^G)$ was obtained by iteratively evaluating RMSE and MAE measures until no considerable gains were obtained by further increasing the number of basis functions.

are decreasing in λ_t if $\psi < 1$, while they could be either increasing or decreasing for $\psi > 1$. Bansal and Yaron (2004) have argued that the IES must be greater than one for the long-run risk model to be able to generate a positive correlation between expected consumption growth and equity prices and at the same time a negative correlation between consumption growth volatility and equity prices. However, our results suggest that those correlations can be obtained in economies subject to disaster risks with arbitrary values of IES as long as agents prefer earlier resolution of uncertainty. From Equation (2.1), the expected consumption growth is given by $\mathbb{E}_t(d \log C_t) = (\mu - \frac{1}{2}\sigma^2 + \lambda_t \mathbb{E}_\zeta(\log Z_t)) \Delta t$. Given that $0 < Z_t \leq 1$, an increase in the disaster intensity lowers the expected consumption growth and at the same time lowers the value of equity. Analogously, a higher probability of disaster increases the volatility of consumption growth, $\text{Var}_t(d \log C_t) = (\sigma^2 + \lambda_t [\mathbb{E}_\zeta(\log Z_t)]^2) \Delta t$, while bringing the equity price down.

In panel (c) we show that the equity risk premium is increasing in λ_t . As opposed to Gourio (2008) and Gourio (2012), the model is able to generate countercyclical equity risk premium and Sharpe Ratios, and a negative correlation between excess returns and the price-dividend ratio for both IES greater and lower than one. The latter can be seen by comparing panels (b) and (c) from Figures (1) and (2) respectively.

Panel (d) shows that the level of volatility in the stock market is positively related to the level of IES. The intuition behind this result becomes clear after looking in detail at Figures (1) and (2). An IES below unity is suggestive of agents being more averse to intertemporal gambles and hence less likely to invest in risky assets. Therefore, equity prices are less responsive to changes in the probability of a disaster given that it is through the risk free rate that the economy mostly adjusts in order to maintain the general equilibrium. The opposite occurs when $\psi > 1$. Since agents invest more of their wealth in the risky asset, more variation in the equity prices are allowed to guarantee the equilibrium of the economy when the disaster intensity changes over time, explaining the higher levels of volatility in the aggregate stock market. As discussed in Wachter (2013), the ability of the model to generate high levels of volatility in stock returns does not compromise its ability to generate low variability in the return on government liabilities. The latter is explained by two competing effects: A higher disaster intensity reduces the return offered by safer assets due to an increased desire to save. But at the same time, an increase in the probability of a disaster leads the agent to demand a higher compensation in order to hold the asset because a higher probability of government default.

4 An empirical assessment of the disaster risk model

This section revisits the fit of the the time-varying disaster risk model in different dimensions by comparing the predictions of the model from simulated data with some stylized facts from the postwar period in the U.S. economy. We choose the postwar period since it allows us to study the virtues and limitations of the time-varying disaster risk model in times where no disasters have occurred. The purpose of the simulation is twofold. By using the exact same calibration as in Wachter (2013), we first reassess the fit of the model using the two-step projection method introduced in Section (3). Second, we evaluate the asset pricing implications of the model for different values of IES while keeping the remaining parameters fixed as in the first experiment.

4.1 Calibration

Table (2) describes the parameter values used to solve and simulate the model. The endowment and disaster intensity processes are calibrated to match salient features of the data set on macroeconomic crises around the world built by Barro and Ursúa (2008). In the data, a macroeconomic disaster refers to declines in aggregate consumption of 10% or more over a given period of time. Using a sample of 22 countries, the average probability of observing such a contraction in aggregate consumption, $\bar{\lambda}$, was found to be 3.55% per year. We approximate the distribution of percentage declines in consumption of 10% or more, $\zeta(Z_t)$, with the empirical distribution (histogram) of such events in the data. The annual consumption growth in normal times in Barro and Ursúa’s data set is 2.52% while its standard deviation is 2.00%. The probability of a partial default by the government is reported to be 0.4 in Barro (2006). Wachter (2013) calibrates the remaining parameters in a way that they replicate the average real return on the 3-month Treasury bill, the autocorrelation of the price-dividend ratio and the volatility of stock returns observed in the data.

[Table 2 about here.]

The simulation results are compared against annual U.S. data for the sample periods 1947-2008 and 1947-2013. We use monthly data on nominal stock returns and aggregate dividends from CRSP⁴. To calculate returns we first construct simple monthly returns, then take logs and finally add the monthly returns within a year. The yearly dividends correspond to the sum of dividends within the year. We then calculate the log year over year growth in dividends. The price-dividend ratio is the price in the last month of the year divided by the sum of dividends paid in the last twelve months. To transform returns and dividend growth to real terms, we use the CPI series from the Bureau of Labor Statistics. Yearly inflation is computed from seasonally unadjusted CPI, and it corresponds to the logarithmic change from December to December. For both dividends and stock returns we subtract log inflation to form real growth rates or returns.

The risk free rate used in the paper is built following Beeler and Campbell (2012). We use the three month nominal yields from the CRSP Fama Risk Free Rates and subtract three month ahead log inflation to initially create a measure of the ex-post real three month real return. We then regress it on the average quarterly log inflation over the previous year, and the three month nominal yield. The predicted value from this regression defines a monthly ex-ante risk free rate. For the expected return on government liabilities we follow Cochrane (2008) and use the nominal 90 day Treasury bill yield from CRSP. We then

⁴Following Cochrane (2008), the stock market returns and aggregated dividends are constructed using the CRSP value-weighted index including (*VWRETD*) and excluding (*VWRETX*) distributions. A monthly price index for the stock market can be constructed as:

$$F_{t+1} = F_t (1 + VWRETX_{t+1}), \quad F_0 = 1.$$

Similarly, a monthly index for dividends is computed according to:

$$D_{t+1} = F_{t+1} \left[\frac{(1 + VWRETD_{t+1})}{(1 + VWRETX_{t+1})} - 1 \right].$$

subtract quarterly inflation to construct a proxy for the real return on partially defaultable government bills.

The consumption data from the Bureau of Economic Analysis shows that the annual average growth rate of per capital real consumption of non durables and services was 1.87% with a standard deviation of 1.32% during the postwar period 1947-2013. If we condition out the Great Recession, the average growth rate increases to 2.00% and its volatility falls to 1.25%.

4.2 Simulation results

To simulate the model, we first discretize the continuous-time state processes in Equations (2.1), (2.2) and (2.24) using a Euler approximation. We then simulate 50,000 years of data at a monthly frequency for the disaster risk λ_t , log consumption growth, Δc_t , and log dividend growth, Δd_t . The technical details regarding the simulation procedure can be found in Glasserman (2003).

Given the simulated values for λ_t , we proceed to compute the consumption-wealth ratio and the price-dividend ratio for each possible value of $\psi \in \{0.5, 1.0, 1.5\}$ using linear interpolation techniques on the approximants constructed from the solution of the model. We use the simulated series for the price-dividend ratio and dividend growth to compute equity returns. Returns on government liabilities are computed using the results from Proposition (2.2), while the risk free rate is computed from Proposition (2.1). Finally, all monthly returns and growth rates are compounded to an annual frequency. In what follows we report two different sets of moments from the simulated data. The first one, under the label “population”, is computed using all the years in the simulation. The second set, under the label “conditional”, is calculated after first removing those years in which one or more disasters occurred during the simulation.

Table (3) reports the sample moments of the simulated process for consumption growth and dividend growth together with those observed in the data. Conditional on no disasters, the stochastic process for the endowment generates an average growth rate and volatility of consumption consistent with the data. The inclusion of disasters in the simulation generates a lower expected consumption growth and a higher volatility reflecting the uncertain times associated with large economic downturns. The jump-diffusion specification in Equation (2.1) can not replicate the autocorrelation observed in the data since, by construction and conditional on no disasters, consumption growth follows a random walk. The inclusion of simulated disasters does not generate further autocorrelation. Given that, by construction, log dividend growth is just a scaled version of log consumption growth, similar conclusions can be drawn from the simulation of aggregate dividends.

[Table 3 about here.]

4.2.1 Revisiting the case of unitary IES

Table (4) reports some basic moments of asset returns from annual U.S. data and from the simulation of the model when the IES equals one. The same exercise was carried out in Wachter (2013) using an approximate closed form solution for the price-dividend ratio.

The overall fit of the time-varying disaster risk model is good. The model generates high levels of equity premium and at the same time produces volatility in the stock market that far exceeds that of dividends without generating counterfactual high volatility of more secure assets like the government liabilities. Conditional on no disasters, the model produces an equity risk premium of 7.97% per year, while this number falls to 6.75% per year when disasters are included in the sample. The average volatility and autocorrelation of price-dividend ratio are all in the vicinity of their data counterparts, while an expected return on government liabilities of 1.36% exactly matches that in the data.

However, the model does not generate as much volatility in the stock market as suggested in Wachter (2013). While still being high relative to other models of asset pricing, we find that when the two step projection method is used the model only produces a standard deviation of stock returns of 11.54% conditional on no jumps, versus 17.7% reported by Wachter. The lower volatility explains the higher levels of the Sharpe Ratio. Therefore, the ability of the time-varying disaster risk model to generate excess volatility is sensible to approximation method used to solve the model and hence any concluding statements in this regards should be taken with caution. As it will be shown later, the level of volatility can be amplified by allowing higher levels of the IES.

[Table 4 about here.]

A deeper look into the results reveals that the model generates highly persistent interest rates. While the autocorrelation of the risk free rate and the returns on government bills has been around 0.43 and 0.55 in the postwar period respectively, conditional on no jumps, the model generates an autocorrelation of 0.94 for both assets. The poor fit comes directly from the fact that all the dynamics in the model are driven by a single state variable, the disaster risk, which has been calibrated to replicate an autocorrelation of 0.92 of the price-dividend ratio. One way to disentangle this issue is to allow for an additional state variable with an autocorrelation level that is negatively related to that of the disaster risk. One alternative could be to include persistence to the expected consumption growth as in the long-run risk literature. By doing so, it will be possible to add persistence to the diffusive component of the stochastic discount factor which will ultimately mitigate the high persistence of the disaster risk that explains both the risk free rate and the expected return on government bonds (see Equations (2.1), (2.16) and (2.19)).

Finally, we find that the model produces a puzzling ranking of volatilities between the risk free rate and the expected return on government liabilities. The model generates the counterfactual implication that the volatility of the risk free rate is higher than that of the government bonds in times of no disasters⁵. While one could argue that this result is consistent with the two opposing effects that explain why the expected return on government bonds is higher than that of the riskless asset, this implication is inconsistent with the risk-return trade-off pillar of the asset pricing literature.

4.2.2 The role of the intertemporal elasticity of substitution

What are the asset pricing implications of lower and higher values of the IES? To address this question we keep the calibration used in the previous section fixed and only change the value of ψ . The correct

⁵Notice that the ranking of volatility is correct for the population moments, where the volatility of government bonds is higher than the volatility of the risk free rate.

value of this parameter is subject to much discussion in the field and it is usually calibrated in such a way that the model delivers implications that are consistent with the data.

The standard view in macroeconomics is that the IES is small or even close to zero when estimated from macroeconomic aggregates (see Hall, 1988). However, the asset pricing literature has challenged this assumption by providing arguments in favor of $\psi > 1$ based on the predictability and countercyclical properties of the equity risk premium (see Bansal and Yaron, 2004, Gourio, 2008 and Gourio, 2012). Macroeconometricians have also found empirical evidence of IES greater than one in more structural macrofinancial models of the DSGE-type that explain the behavior of the term structure of interest rates (see van Binsbergen et al., 2012)⁶.

To understand the role played by the IES, suppose for a moment that the economy is hit by a negative shock to the disaster risk, i.e. it experiences a reduction in the probability of a disaster. The lower is the likelihood of a catastrophe, the higher is the expected growth of consumption which materializes through a change in the level of the consumption-wealth ratio. Since consumption is exogenously given in this economy, all the adjustment is done through the reallocation of wealth. The direction of the adjustment depends on the level of the IES. If the substitution effect dominates the wealth effect, i.e. if $\psi > 1$, then the representative investor would demand more of the risky asset bringing its price up. Otherwise, if the wealth effect dominates the substitution effect, i.e. if $\psi < 1$, then the agent prefers to consume more today driving the prices of the risky asset down.

Table (5) reports sample moments from simulated data of the model for different values of the IES conditional on no jumps under the assumption that the representative agent prefers early resolution of uncertainty. The results suggest that the expected return on government bills and its volatility are decreasing in the IES. We also find that the expected equity returns are decreasing in the IES, while the stock market volatility is increasing in the IES. The autocorrelation of the equity return and of the risk free rate are independent of the IES. The same holds for the government bills. The overall effect on the equity risk premium is dominated by the movements in the expected returns on government bills which lead to a premium that is decreasing in the IES.

[Table 5 about here.]

When compared to the the postwar data, the calibration used in this paper favors an IES below unity since it brings the model's equity risk premium from 7.97% to 6.34% per year, closer to the 5.11% observed in the data. A IES below unity is also able to generate a more volatile return on government bonds, closer to the postwar period levels. On the other hand, a model where $\psi > 1$ is needed to account for the large volatility in the stock market and the volatility of the price-dividend ratio, which ultimately will lead to a lower Sharpe Ratio.

⁶However, some results in favor of $\psi < 1$ coming from the asset pricing literature have been documented recently when the model incorporates limited asset market participation. See Vissing-Jørgensen (2002) and Guvenen (2006).

4.3 Predictability

Contrary to the implications of the constant probability model of Barro (2006), the time-varying disaster risk model is able to generate time variation in the valuation ratios as well as in the market returns. Thus, it is natural to assess the ability of the model to match some of the empirical facts regarding excess return, consumption growth and dividend growth predictability. It is important to keep in mind that all the time variation in the model comes directly from the assumptions on the exogenous process for the disaster risk.

Empirical evidence suggests that both the stock returns and the excess returns are forecastable in long horizons. It has also been argued that equity returns are predictable, because the risk premium are forecastable, and not because the risk free interest rate is forecastable (see Cochrane, 2008 and Gourio, 2008). More important is the fact that the explanatory power of valuation ratios to predict the risk premium, measured by standard R^2 , increases with the forecast horizon as documented in Fama and French (1988). Therefore, as argued in Cochrane (2008), given the lack of evidence on price-dividend ratios predicting dividend growth, it must be the case that all variation in the price-dividend ratio is due to changes in risk premium and none to changes in expected dividend growth. Given the close link between variation in dividend and consumption growth with future expected changes in returns we also test the ability the price-dividend ratio to predict consumption and dividend growth.

We test the forecasting ability of the model by regressing simulated excess equity returns (log return on equity minus log return on the government bill), consumption growth and dividend growth measured over horizons of 1, 3 and 5 years, onto the simulated log price-dividend ratio. We run all the regressions in both the population sample and the conditional sample. We then compare these results to those obtained by performing the same regressions with observed data from the postwar period.

[Table 6 about here.]

Table (6) indicates that in the data the price-dividend ratio predicts excess returns with a negative sign, and its predictive power is increasing with the time horizon. In fact, its forecast ability increases from 10% at one year horizon to 26% at a 5 year horizon. Consistent with the results from Section (3), the model's implied price-dividend ratio predicts a negative correlation with excess returns given that a high valuation ratio can only be obtained by a low probability of a disaster, which at the same time explains a low equity risk premium. Conditional on no jumps, the model also generates an increasing predictability of the price-dividend ratio, close to that observed in the data. As argued in Wachter (2013), the R^2 measures from the simulation that include disasters are much lower since most of the variation in the equity risk premium is due to changes in the dividend growth after a disaster occurs and not in the price-dividend ratio. Notice that for large IES values the model's R^2 is lower and closer to the data. For these parameters, a higher IES is needed to reduce the excess predictability of the price-dividend ratio.

[Table 7 about here.]

Tables (7) reports regression results for consumption growth. Similar results are obtained for dividend

growth since in the model they are just an affine transformation of consumption⁷. Recall that the time-varying disaster risk model tries to explain the variability of asset returns by assuming large but infrequent drops in aggregate consumption. Panel (C) indicates that there is little predictability of consumption growth at long horizons during the postwar period. Similar results have been found in Hall (1988), Cochrane (1994) and Campbell (2003). The results displayed in Panel (B) are consistent with the fact that conditional on no disasters, both consumption and dividend growth follow random walks and hence are not predictable. When we include realized disasters in the simulation the model generates predictability in consumption growth that exceeds that observed in the data. However, this excess, which comes entirely from the occurrence of a disaster, is not big and never rises above 4.8%. Regarding the role played by the IES, this parameter does not compromise the predictability properties of the model, even though the relative effect of the price-dividend ratio on both consumption and dividend growth rates is decreasing in ψ .

Our results suggest that the time-varying disaster risk model produces consistent implications in terms of predictability as opposed to those from the long-run risk framework of Bansal and Yaron (2004). As shown in Beeler and Campbell (2012), their model generates predictability for consumption and dividend growth that far exceeds that found in the data.

5 Conclusions

Time-varying disaster risk models have the potential to reconcile several asset pricing puzzles in a single unified framework. The first models of this type focused on a constant probability of disaster aiming at explaining the equity premium puzzle. The latest generation of models incorporate time-varying disaster probabilities (See Gourio, 2012 and Wachter, 2013) which allow them to improve the performance of the model by explaining other asset pricing puzzles.

In this paper we revisit the fit of time-varying disaster risk models to the postwar U.S. data. Our results confirm the virtues of the model and reveals some of its limitations. We calibrate the model using standard values and compare the model simulations with those from the data conditional on no crisis. Crucially, we allow the elasticity of intertemporal substitution parameter to accommodate different values.

For the case of unitary IES, we find that the approximation method is decisive to generate aggregate stock market volatility close to that observed in the data. Whereas the approximated closed form solution of Wachter (2013) generates a standard deviation of stock returns of around 17.7 percent per year, the two step projection method proposed in this paper only produces an annual volatility of 11.5 percent.

We also find that the model produces near unit root risk free and expected government bill returns which is at odds with the U.S. postwar data. The model also generates the counterfactual implication that the volatility of the risk free rate is higher than that on the returns of Treasury bills.

Our numerical experiments show that opposite to previous results in the literature, the predictability properties of the time-varying disaster risk model do not require the IES to be greater or equal to one.

⁷We omit them because of space considerations but are available upon request.

In fact, for reasonable parameters, we always find a negative correlation between the price-dividend ratio and excess returns. These results suggest that a low IES is consistent with a countercyclical equity risk premium as long as the representative agent prefers an early resolution of uncertainty. In general, we find that an IES lower than one has the potential effect of generating appropriate levels of equity risk premium and volatility of government bonds, while an IES greater than one can generate higher price-dividend volatility and stock market volatility, and hence a lower Sharpe Ratio.

A Appendix

A.1 Homogeneity of the value function

Lemma 1 (Homogeneity of the value function). *Let V_t denote the indirect utility associated with a stream of consumption $\{C_s\}_{s=t}^{\infty}$. Now let $C^\kappa = \{\kappa C_s\}_{s=t}^{\infty}$. Indirect utility V_s^κ associated with C^κ is $V_s^\kappa = \kappa^{1-\gamma} V_s$.*

Proof. We first establish that the aggregator satisfies $f(\kappa C_s, \kappa^{1-\gamma} V_s) = \kappa^{1-\gamma} f(C_s, V_s)$. If $\psi = 1$ then

$$\begin{aligned} f(\kappa C_s, \kappa^{1-\gamma} V_s) &= \beta(1-\gamma) \kappa^{1-\gamma} V_s \left(\log \kappa C_s - \frac{1}{1-\gamma} \log((1-\gamma) \kappa^{1-\gamma} V_s) \right) \\ &= \kappa^{1-\gamma} \beta(1-\gamma) V_s \left(\log C_s - \frac{1}{1-\gamma} \log((1-\gamma) V_s) \right) \\ &= \kappa^{1-\gamma} f(C_s, V_s). \end{aligned}$$

If $\psi \neq 1$ then

$$\begin{aligned} f(\kappa C_s, \kappa^{1-\gamma} V_s) &= \beta \theta \kappa^{1-\gamma} V_s \left[\left(\frac{\kappa C_s}{((1-\gamma) \kappa^{1-\gamma} V_s)^{\frac{1}{1-\gamma}}} \right)^{1-\frac{1}{\psi}} - 1 \right] \\ &= \kappa^{1-\gamma} \beta \theta V_s \left[\left(\frac{C_s}{((1-\gamma) V_s)^{\frac{1}{1-\gamma}}} \right)^{1-\frac{1}{\psi}} - 1 \right] \\ &= \kappa^{1-\gamma} f(C_s, V_s). \end{aligned}$$

Hence,

$$\begin{aligned} V_t^\kappa &= \mathbb{E}_t \left[\int_t^\infty f(\kappa C_s, V_s^\kappa) ds \right] \\ &= \mathbb{E}_t \left[\int_t^\infty f(\kappa C_s, \kappa^{1-\gamma} V_s) ds \right] \\ &= \kappa^{1-\gamma} \mathbb{E}_t \left[\int_t^\infty f(C_s, V_s) ds \right] \\ &= \kappa^{1-\gamma} V_t. \end{aligned}$$

□

A.2 Hamilton-Jacobi-Bellman Equation

The Hamilton-Jacobi-Bellman equation (HJB) for the planner's problem is given by:

$$0 = f(C_t, V_t) + \mu C_t V_C + \kappa (\bar{\lambda} - \lambda_t) V_\lambda + \frac{1}{2} C_t^2 \sigma^2 V_{CC} + \frac{1}{2} \sigma_\lambda^2 \lambda_t V_{\lambda\lambda} + \lambda_t \mathbb{E}_\zeta (V(Z_t C_t, \lambda_t) - V(C_t, \lambda_t)).$$

We now conjecture that the value function $V(C, \lambda)$ takes the form:

$$V_t = \frac{1}{1-\gamma} C_t^{1-\gamma} H(\lambda_t), \quad (C, \lambda) \in (0, \infty) \times \mathbb{R}_+.$$

Substitute the conjecture back into the HJB equation to obtain:

$$\begin{aligned} 0 = f(C_t, V_t) + \mu C_t^{1-\gamma} H(\lambda_t) + \kappa (\bar{\lambda} - \lambda_t) \frac{1}{1-\gamma} C_t^{1-\gamma} H_\lambda(\lambda_t) - \frac{1}{2} \sigma^2 C_t^{1-\gamma} H(\lambda_t) \\ + \frac{1}{2} \sigma_\lambda^2 \lambda_t \frac{1}{1-\gamma} C_t^{1-\gamma} H_{\lambda\lambda}(\lambda_t) + \lambda_t \frac{1}{1-\gamma} C_t^{1-\gamma} H_\lambda(\lambda_t) \mathbb{E}_\zeta (Z_t^{1-\gamma} - 1). \end{aligned}$$

Dividing by V_t yields:

$$0 = \frac{f(C_t, V_t)}{V_t} + (1-\gamma)\mu + \kappa (\bar{\lambda} - \lambda_t) \frac{H_\lambda(\lambda_t)}{H(\lambda_t)} - \frac{1}{2} \gamma (1-\gamma) \sigma^2 + \frac{1}{2} \sigma_\lambda^2 \lambda_t \frac{H_{\lambda\lambda}(\lambda_t)}{H(\lambda_t)} + \lambda_t \mathbb{E}_\zeta (Z_t^{1-\gamma} - 1),$$

where

$$\frac{f(C_t, V_t)}{V_t} = \begin{cases} \beta\theta \left[H(\lambda_t)^{-\frac{1}{\theta}} - 1 \right], & \text{if } \psi \neq 1 \\ -\beta \log H(\lambda_t), & \text{if } \psi = 1 \end{cases}$$

and $\theta = \frac{1-\gamma}{1-\psi}$. Reorganizing terms results in a quasi-linear PDE of the form:

$$\begin{aligned} 0 = \frac{f(C_t, V_t)}{V_t} H(\lambda_t) + \left[(1-\gamma)\mu - \frac{1}{2} \gamma (1-\gamma) \sigma^2 + \lambda_t \mathbb{E}_\zeta (Z_t^{1-\gamma} - 1) \right] H(\lambda_t) \\ + \kappa (\bar{\lambda} - \lambda_t) H_\lambda(\lambda_t) + \frac{1}{2} \sigma_\lambda^2 \lambda_t H_{\lambda\lambda}(\lambda_t), \end{aligned}$$

where the non-linear term is given by:

$$\frac{f(C_t, V_t)}{V_t} H(\lambda_t) = \begin{cases} \frac{\beta}{1-q} H(\lambda_t)^q - \frac{\beta}{1-q} H(\lambda_t) & \text{if } \psi \neq 1 \\ -\beta H(\lambda_t) \log H(\lambda_t) & \text{if } \psi = 1 \end{cases}$$

and $q = 1 - \frac{1}{\theta}$.

A.3 Stochastic discount factor and risk free rate

From the definition of the utility aggregator in Equations (2.4) and (2.5) we can compute:

$$f_C(C_t, V_t) = \begin{cases} \beta C_t^{-\gamma} H(\lambda_t)^q, & \text{if } \psi \neq 1 \\ \beta C_t^{-\gamma} H(\lambda_t), & \text{if } \psi = 1 \end{cases}$$

and

$$f_V(C_t, V_t) = \begin{cases} \frac{\beta}{1-q} [qH(\lambda_t)^{q-1} - 1], & \text{if } \psi \neq 1 \\ -\beta(1 + \log H(\lambda_t)), & \text{if } \psi = 1. \end{cases}$$

Itô's lemma combined with the Leibniz rule yields the following dynamics for the SDF in Equation (2.11):

$$\frac{d\pi_t}{\pi_t} = f_V(C_t, V_t) dt + \frac{df_C(C_t, V_t)}{f_C(C_t, V_t)}.$$

where the second term on the right hand side is given by:

$$\begin{aligned} \frac{df_C}{f_C} = & \left(-\gamma\mu + \frac{1}{2}\gamma(1+\gamma)\sigma^2 + \kappa(\bar{\lambda} - \lambda_t)q \frac{H_\lambda(\lambda_t)}{H(\lambda_t)} + \frac{1}{2}q \left((q-1) \left(\frac{H_\lambda(\lambda_t)}{H(\lambda_t)} \right)^2 + \frac{H_{\lambda\lambda}(\lambda_t)}{H(\lambda_t)} \right) \sigma_\lambda^2 \lambda_t \right) dt \\ & - \gamma\sigma dB_t + q \frac{H_\lambda(\lambda_t)}{H(\lambda_t)} \sigma_\lambda \sqrt{\lambda_t} dB_{\lambda,t} + (Z_t^{-\gamma} - 1) dN_t \end{aligned}$$

when $\psi \neq 1$ and by

$$\begin{aligned} \frac{df_C}{f_C} = & \left(-\gamma\mu + \frac{1}{2}\gamma(1+\gamma)\sigma^2 + \kappa(\bar{\lambda} - \lambda) \frac{H_\lambda(\lambda_t)}{H(\lambda_t)} + \frac{1}{2} \frac{H_{\lambda\lambda}(\lambda_t)}{H(\lambda_t)} \sigma_\lambda^2 \lambda^2 \right) dt \\ & - \gamma\sigma dB_t + \frac{H_\lambda(\lambda_t)}{H(\lambda_t)} \sigma_\lambda \sqrt{\lambda_t} dB_{\lambda,t} + (Z_t^{-\gamma} - 1) dN_t \end{aligned}$$

when $\psi = 1$.

Therefore:

$$\frac{d\pi_t}{\pi_t} = \mu_{\pi,t} dt + \boldsymbol{\sigma}_{\pi,t} \left[\begin{matrix} dB_t & dB_{\lambda,t} \end{matrix} \right]' + (Z_t^{-\gamma} - 1) dN_t$$

where

$$\mu_{\pi,t} = \begin{cases} -\frac{\beta}{1-q} - \gamma\mu + \frac{1}{2}\gamma(1+\gamma)\sigma^2 \\ \quad + q \left[\frac{\beta}{1-q} H(\lambda_t)^{q-1} + \kappa(\bar{\lambda} - \lambda_t) \frac{H_{\lambda}(\lambda_t)}{H(\lambda_t)} + \frac{1}{2} \left((q-1) \left(\frac{H_{\lambda}(\lambda_t)}{H(\lambda_t)} \right)^2 + \frac{H_{\lambda\lambda}(\lambda_t)}{H(\lambda_t)} \right) \sigma_{\lambda}^2 \lambda_t \right] & \text{if } \psi \neq 1 \\ -\beta - \gamma\mu + \frac{1}{2}\gamma(1+\gamma)\sigma^2 - \beta \log H(\lambda_t) + \kappa(\bar{\lambda} - \lambda) \frac{H_{\lambda}(\lambda_t)}{H(\lambda_t)} + \frac{1}{2} \frac{H_{\lambda\lambda}(\lambda_t)}{H(\lambda_t)} \sigma_{\lambda}^2 \lambda_t & \text{if } \psi = 1 \end{cases}$$

and

$$\sigma_{\pi,t} = \begin{cases} \left[-\gamma\sigma \quad q \frac{H_{\lambda}(\lambda_t)}{H(\lambda_t)} \sigma_{\lambda} \sqrt{\lambda_t} \right] & \text{if } \psi \neq 1 \\ \left[-\gamma\sigma \quad \frac{H_{\lambda}(\lambda_t)}{H(\lambda_t)} \sigma_{\lambda} \sqrt{\lambda_t} \right] & \text{if } \psi = 1. \end{cases}$$

The basic asset pricing Equation (2.10) implies that for an asset that has a constant price of one and pays the risk free rate, r_t^f , as dividend the following holds:

$$\begin{aligned} r_t^f &= -\frac{1}{dt} \mathbb{E}_t \left(\frac{d\pi_t}{\pi_t} \right) \\ &= -\mu_{\pi,t} - \lambda_t \mathbb{E}_{\zeta} (Z_t^{-\gamma} - 1). \end{aligned}$$

A.4 Return on short-term government liabilities

The price of a government liability satisfies (dividend are zero):

$$\begin{aligned} 0 &= \frac{1}{dt} \mathbb{E}_t [d(\pi_t L_t)] \\ &= r_t^L + \mu_{\pi,t} + \lambda_t \mathbb{E}_{\zeta} [Z_t^{-\gamma} Z_{L,t} - 1] \end{aligned}$$

From the definition of $Z_{L,t}$ it follows that:

$$\mathbb{E}_{\zeta} [Z_t^{-\gamma} Z_{L,t} - 1] = p \mathbb{E}_{\zeta} [Z_t^{1-\gamma} - 1] + (1-p) \mathbb{E}_{\zeta} [Z_t^{-\gamma} - 1].$$

Hence,

$$\begin{aligned} r_t^L &= -\mu_{\pi,t} - \lambda_t \left[p \mathbb{E}_{\zeta} [Z_t^{1-\gamma} - 1] + (1-p) \mathbb{E}_{\zeta} [Z_t^{-\gamma} - 1] \right] \\ r_t^L &= r_t^f + \lambda_t p \mathbb{E}_{\zeta} [Z_t^{-\gamma} (1 - Z_t)] \end{aligned}$$

Let r_t^b denote the instantaneous expected return on government debt. Then:

$$\begin{aligned}
r_t^b &= \frac{1}{dt} \mathbb{E}_t \left(\frac{dL_t}{L_t} \right) \\
&= r_t^L + \lambda_t p \mathbb{E}_\zeta (Z_t - 1) \\
&= r_t^f + \lambda_t p \mathbb{E}_\zeta [Z_t^{-\gamma} (1 - Z_t)] + \lambda_t p \mathbb{E}_\zeta (Z_t - 1) \\
&= r_t^f + \lambda_t p \mathbb{E}_\zeta [Z_t^{-\gamma} - Z_t^{1-\gamma} + Z_t - 1] \\
&= r_t^f + \lambda_t p \mathbb{E}_\zeta [(Z_t^{-\gamma} - 1)(1 - Z_t)].
\end{aligned}$$

A.5 Wealth-consumption ratio and consumption risk premium

Recall that

$$\pi_t = \exp \left\{ \frac{\beta}{1-q} \int_0^t [qH(\lambda_s)^{q-1} - 1] ds \right\} \beta C_t^{-\gamma} H(\lambda_t)^q,$$

and

$$W_t = S_t = \begin{cases} \frac{1}{\beta} C_t H(\lambda_t; \psi)^{1-q} & \text{if } \psi \neq 1 \\ \frac{1}{\beta} C_t & \text{if } \psi = 1. \end{cases}$$

Therefore:

$$\pi_t S_t = \exp \left\{ \frac{\beta}{1-q} \int_0^t [qH(\lambda_s)^{q-1} - 1] ds \right\} C_t^{1-\gamma} H(\lambda_t).$$

The value function for the representative agent is given by:

$$V_t = \frac{1}{1-\gamma} C_t^{1-\gamma} H(\lambda_t),$$

and hence:

$$\pi_t S_t = (1-\gamma) \exp \left\{ \frac{\beta}{1-q} \int_0^t [qH(\lambda_s)^{q-1} - 1] ds \right\} V_t.$$

Itô's lemma yields:

$$\begin{aligned}
d(\pi_t S_t) &= (1-\gamma) \left[\exp \left\{ \frac{\beta}{1-q} \int_0^t [qH(\lambda_s)^{q-1} - 1] ds \right\} dV_t \right. \\
&\quad \left. + V_t \exp \left\{ \frac{\beta}{1-q} \int_0^t [qH(\lambda_s)^{q-1} - 1] ds \right\} \left(-\frac{\beta}{1-q} + \frac{\beta}{1-q} qH(\lambda_t)^{q-1} \right) dt \right],
\end{aligned}$$

which can be simplified to:

$$d(\pi_t S_t) = (1 - \gamma) \exp \left\{ \frac{\beta}{1 - q} \int_0^t [qH(\lambda_s)^{q-1} - 1] ds \right\} \left[dV_t - V_t \left(\frac{\beta}{1 - q} - \frac{\beta}{1 - q} qH(\lambda_t)^{q-1} \right) dt \right].$$

From the definition of recursive preferences we know that $V(C_t, \lambda_t) + \int_0^t f(C_s, V(C_s, \lambda_s)) ds$ is a martingale. Therefore:

$$\mathbb{E}_t [dV(C_t, \lambda_t) + f(C_t, V(C_t, \lambda_t)) dt] = 0,$$

and hence:

$$dV_t = -f(C_t, V_t) dt + dM_t$$

for some P -martingale M_t . Now recall that:

$$\frac{f(C_t, V_t)}{V_t} = \frac{\beta}{1 - q} [H(\lambda_t)^{q-1} - 1]$$

and thus:

$$dV_t = -V_t \frac{\beta}{1 - q} [H(\lambda_t)^{q-1} - 1] dt + dM_t.$$

Substituting back yields:

$$\begin{aligned} d(\pi_t S_t) &= (1 - \gamma) \exp \left\{ \frac{\beta}{1 - q} \int_0^t [qH(\lambda_s)^{q-1} - 1] ds \right\} \\ &\quad \left[-V_t \frac{\beta}{1 - q} [H(\lambda_t)^{q-1} - 1] dt + dM_t - V_t \left(\frac{\beta}{1 - q} - \frac{\beta}{1 - q} qH(\lambda_t)^{q-1} \right) dt \right] \\ &= (1 - \gamma) \exp \left\{ \frac{\beta}{1 - q} \int_0^t [qH(\lambda_s)^{q-1} - 1] ds \right\} \left[-\frac{\beta}{1 - q} V_t H(\lambda_t)^{q-1} (1 - q) dt + dM_t \right] \\ &= \exp \left\{ \frac{\beta}{1 - q} \int_0^t [qH(\lambda_s)^{q-1} - 1] ds \right\} [-C_t \beta C_t^{-\gamma} H(\lambda_t)^q dt + (1 - \gamma) dM_t] \\ &= -\pi_t C_t dt + \exp \left\{ \frac{\beta}{1 - q} \int_0^t [qH(\lambda_s)^{q-1} - 1] ds \right\} (1 - \gamma) dM_t \end{aligned}$$

where we have used the fact that:

$$V_t H(\lambda_t)^{q-1} = \frac{1}{1 - \gamma} C_t^{1-\gamma} H(\lambda_t)^q.$$

Now integrate from t to T to obtain:

$$\pi_T S_T - \pi_t S_t = - \int_t^T \pi_s C_s ds + (1 - \gamma) \int_t^T \exp \left\{ \frac{\beta}{1 - q} \int_0^t [qH(\lambda_s)^{q-1} - 1] ds \right\} dM_t.$$

Taking expectations, letting $T \rightarrow \infty$, and assuming that the transversality condition, $\lim_{T \rightarrow \infty} \mathbb{E} [\pi_T S_T] = 0$:

$$\lim_{T \rightarrow \infty} \mathbb{E}_t [\pi_T S_T] - \mathbb{E}_t [\pi_t S_t] = - \mathbb{E}_t \left[\int_t^\infty \pi_s C_s ds \right] + (1 - \gamma) \mathbb{E}_t \left[\int_t^\infty \exp \left\{ \frac{\beta}{1 - q} \int_0^t [qH(\lambda_s)^{q-1} - 1] ds \right\} dM_t \right]$$

Therefore:

$$\pi_t S_t = \mathbb{E}_t \left[\int_t^\infty \pi_s C_s ds \right].$$

This shows that the wealth-consumption ratio in Equations (2.20) and (2.21) satisfy the basic asset pricing equation.

Itô's lemma implies that the dynamics of the price of a consumption claim is given by:

$$\frac{dS_t}{S_t} = \mu_{S,t} dt + \sigma_{S,t} \left[dB_t \quad dB_{\lambda,t} \right]' + (Z_t - 1) dN_t$$

where

$$\mu_{S,t} = \mu + (1 - q) \frac{H_\lambda(\lambda_t)}{H(\lambda_t)} \kappa (\bar{\lambda} - \lambda_t) + \frac{1}{2} (1 - q) \left(-q \left(\frac{H_\lambda(\lambda_t)}{H(\lambda_t)} \right)^2 + \frac{H_{\lambda\lambda}(\lambda_t)}{H(\lambda_t)} \right) \sigma_\lambda^2 \lambda_t$$

and

$$\sigma_{S,t} = \left[\sigma \quad (1 - q) \frac{H_\lambda(\lambda_t)}{H(\lambda_t)} \sigma_\lambda \sqrt{\lambda_t} \right].$$

The risk premium on the aggregate consumption claim is derived from the Martingale condition (The basic asset pricing equation for risky assets):

$$\mathbb{E}_t \left[\frac{d(\pi_t S_t)}{\pi_t S_t} \right] + \frac{C_t}{S_t} dt = 0.$$

Itô's lemma implies:

$$\begin{aligned} \frac{1}{dt} \mathbb{E}_t \left[\frac{d(\pi_t S_t)}{\pi_t S_t} \right] &= \frac{1}{dt} \mathbb{E}_t \left[\frac{d\pi_t}{\pi_t} + \frac{dS_t}{S_t} + \frac{d\pi_t}{\pi_t} \frac{dS_t}{S_t} \right] \\ &= -r_t^f + \frac{1}{dt} \mathbb{E}_t \left[\frac{dS_t}{S_t} + \frac{d\pi_t}{\pi_t} \frac{dS_t}{S_t} \right]. \end{aligned}$$

Hence, the consumption risk premium, rp_t^C , can be computed from:

$$\underbrace{\frac{1}{dt} \mathbb{E}_t \left[\frac{dS_t}{S_t} \right] + \frac{C_t}{S_t} dt - r_t^f}_{:=rp_t^C} = -\frac{1}{dt} \mathbb{E}_t \left[\frac{d\pi_t}{\pi_t} \frac{dS_t}{S_t} \right]$$

where the term on the right hand side is the covariance of the asset return with the SDF. In particular:

$$\begin{aligned} rp_t^C &= -\frac{1}{dt} \mathbb{E}_t \left[(\boldsymbol{\sigma}_{\pi,t} \boldsymbol{\sigma}'_{S,t} dt + (Z_t^{-\gamma} - 1) (Z_t - 1) dN_t) \right] \\ &= -\boldsymbol{\sigma}_{\pi,t} \boldsymbol{\sigma}'_{S,t} - \lambda_t \mathbb{E}_\zeta \left[(Z_t^{-\gamma} - 1) (Z_t - 1) \right] \\ &= \gamma \sigma^2 - q(1 - q) \left(\frac{H_\lambda(\lambda_t)}{H(\lambda_t)} \right)^2 \sigma_\lambda^2 \lambda_t + \lambda_t \mathbb{E}_\zeta \left[(Z_t^{-\gamma} - 1) (1 - Z_t) \right] \end{aligned}$$

The expression for the equity premium just derived is inclusive of jumps as can be seen from the definition of rp_t^C . More specifically, note that:

$$\begin{aligned} rp_t^C &= \frac{1}{dt} \mathbb{E}_t \left[\frac{dS_t}{S_t} \right] + \frac{C_t}{S_t} dt - r_t \\ &= \mu_{S,t} + \lambda_t \mathbb{E}_\zeta [Z_t - 1]. \end{aligned}$$

Hence, the risk premium conditional on no disasters, which we denote by $\tilde{r}p_t^C$, is given by:

$$\tilde{r}p_t^C = rp_t^C - \lambda_t \mathbb{E}_\zeta [Z_t - 1].$$

A.6 Price-dividend ratio and equity risk premium

Using the definition of the log-price dividend ratio in Equation (2.26), Itô's lemma implies:

$$\frac{dF_t}{F_t} = \mu_{F,t} dt + \boldsymbol{\sigma}_{F,t} \left[\begin{array}{cc} dB_t & dB_{\lambda,t} \end{array} \right]' - (1 - Z_t^\varphi) dN_t.$$

where

$$\mu_{F,t}(\lambda_t; \psi) = \mu_D + G_\lambda(\lambda_t; \psi) \kappa (\bar{\lambda} - \lambda_t) + \frac{1}{2} \sigma_\lambda^2 \lambda_t \left(G_\lambda(\lambda_t; \psi)^2 + G_{\lambda\lambda}(\lambda_t; \psi) \right)$$

and

$$\boldsymbol{\sigma}_{F,t} = \left[\begin{array}{cc} \varphi \sigma & G_\lambda(\lambda_t; \psi) \sigma_\lambda \sqrt{\lambda_t} \end{array} \right].$$

Recall that the price of a claim to future dividends can be written as:

$$\pi_t F_t = \mathbb{E}_t \left[\int_t^\infty \pi_s D_s ds \right]$$

which also holds for $u > t$:

$$\pi_s F_s = \mathbb{E}_s \left[\int_s^\infty \pi_s D_s ds \right].$$

Hence:

$$\begin{aligned} \pi_t F_t &= \mathbb{E}_t \left[\int_t^\infty \pi_s D_s ds \right] \\ &= \mathbb{E}_t \left[\int_t^u \pi_s D_s ds + \int_u^\infty \pi_s D_s ds \right] \\ &= \mathbb{E}_t \left[\pi_u F_u + \int_t^u \pi_s D_s ds \right]. \end{aligned}$$

Now add and subtract $\int_0^t \pi_s D_s ds$:

$$\begin{aligned} \pi_t F_t + \int_0^t \pi_s D_s ds &= \mathbb{E}_t \left[\pi_u F_u + \int_0^t \pi_s D_s ds + \int_t^u \pi_s D_s ds \right] \\ \pi_t F_t + \int_0^t \pi_s D_s ds &= \mathbb{E}_t \left[\pi_u F_u + \int_0^u \pi_s D_s ds \right], \end{aligned}$$

which shows that $\pi_t F_t + \int_0^t \pi_s D_s ds$ is a martingale. Now apply Itô's lemma to the product $\pi_t F_t$ and integrate in the interval $[0, t]$:

$$\begin{aligned} \pi_t F_t - \pi_0 F_0 &= \int_0^t \pi_s F_s (\mu_{\pi,s} + \mu_{F,s} + \sigma_{\pi,s} \sigma'_{F,s}) ds \\ &\quad + \int_0^t \pi_s F_s (\sigma_{\pi,s} + \sigma_{F,s}) \left[dB_s \quad dB_{\lambda,s} \right]' + \int_0^t \pi_s F_s (Z_s^{(\varphi-\gamma)} - 1) dN_s. \end{aligned}$$

We now add and subtract the term $\int_0^t \pi_s D_s ds$ to obtain:

$$\begin{aligned} \pi_t F_t + \int_0^t \pi_s D_s ds - \pi_0 F_0 &= \int_0^t \pi_s F_s \left(\mu_{\pi,t} + \mu_{F,s} + \frac{D_s}{F_s} + \sigma_{\pi,s} \sigma'_{F,s} \right) ds \\ &\quad + \int_0^t \pi_s F_s (\sigma_{\pi,s} + \sigma_{F,s}) \left[dB_s \quad dB_{\lambda,s} \right] + \int_0^t \pi_s F_s \left(Z_s^{(\varphi-\gamma)} - 1 \right) dN_s, \end{aligned}$$

By noticing that the mean of the jump component is given by $\lambda_t \mathbb{E}_\zeta \left(Z_t^{(\varphi-\gamma)} - 1 \right) dt$, we add and subtract the compensation term $\int_0^t \pi_s F_s \lambda_s \mathbb{E}_\zeta \left(Z_s^{(\varphi-\gamma)} - 1 \right) ds$ to obtain:

$$\begin{aligned} \pi_t F_t + \int_0^t \pi_s D_s ds - \pi_0 F_0 &= \int_0^t \pi_s F_s \left(\mu_{\pi,s} + \mu_{F,s} + \frac{D_s}{F_s} + \sigma_{\pi,s} \sigma'_{F,s} + \lambda_s \mathbb{E}_\zeta \left(Z_s^{(\varphi-\gamma)} - 1 \right) \right) ds \\ &\quad + \int_0^t \pi_s F_s (\sigma_{\pi,s} + \sigma_{F,s}) \left[dB_s \quad dB_{\lambda,s} \right]' + \left(\int_0^t \pi_s F_s \left(Z_s^{(\varphi-\gamma)} - 1 \right) dN_s - \int_0^t \pi_s F_s \lambda_s \mathbb{E}_\zeta \left(Z_s^{(\varphi-\gamma)} - 1 \right) ds \right). \end{aligned}$$

Using the properties of stochastic integrals, the second and third terms on the right hand side are martingales. Since the term on the left hand side is a martingale, then the first term on the right hand side must be also a martingale implying that:

$$\mu_{\pi,t} + \mu_{F,t} + \frac{D_t}{F_t} + \sigma_{\pi,t} \sigma'_{F,t} + \lambda_t \mathbb{E}_\zeta \left(Z_t^{(\varphi-\gamma)} - 1 \right) = 0$$

which reduces to:

$$-r_t^f - \lambda_t \mathbb{E}_\zeta \left(Z_t^{-\gamma} - 1 \right) + \mu_{F,t} + \frac{D_t}{F_t} + \sigma_{\pi,t} \sigma'_{F,t} + \lambda_t \mathbb{E}_\zeta \left(Z_t^{(\varphi-\gamma)} - 1 \right) = 0.$$

Using the definition of the price-dividend ratio yields:

$$\begin{aligned} \frac{1}{\exp(G(\lambda_t))} - r_t + \lambda_t \mathbb{E}_\zeta \left(Z_t^{(\varphi-\gamma)} - Z_t^{-\gamma} \right) + \varphi \mu + \frac{1}{2} \sigma^2 \varphi (\varphi - 1 - 2\gamma) \\ + \left(\kappa (\bar{\lambda} - \lambda_t) + q \frac{H_\lambda(\lambda_t)}{H(\lambda_t)} \sigma_\lambda^2 \lambda_t \right) G_\lambda(\lambda_t) + \frac{1}{2} \sigma_\lambda^2 \lambda_t G_{\lambda\lambda}(\lambda_t) + \frac{1}{2} \sigma_\lambda^2 \lambda_t G_\lambda(\lambda_t)^2 = 0. \end{aligned}$$

The risk premium on the aggregate dividend claim is derived from the Martingale condition:

$$\mathbb{E}_t \left[\frac{d(\pi_t F_t)}{\pi_t F_t} \right] + \frac{D_t}{F_t} = 0.$$

Itô's lemma implies:

$$\begin{aligned}
\frac{1}{dt} \mathbb{E}_t \left[\frac{d(\pi_t F_t)}{\pi_t F_t} \right] &= \frac{1}{dt} \mathbb{E}_t \left[\frac{d\pi_t}{\pi_t} + \frac{dF_t}{F_t} + \frac{d\pi_t}{\pi_t} \frac{dF_t}{F_t} \right] \\
&= -r_t^f + \frac{1}{dt} \mathbb{E}_t \left[\frac{dF_t}{F_t} + \frac{d\pi_t}{\pi_t} \frac{dF_t}{F_t} \right].
\end{aligned}$$

Hence the equity risk premium, rp_t^D , can be computed from:

$$\underbrace{\frac{1}{dt} \mathbb{E}_t \left[\frac{dF_t}{F_t} \right] + \frac{D_t}{F_t} - r_t^f}_{:=rp_t^D} = -\frac{1}{dt} \mathbb{E}_t \left[\frac{d\pi_t}{\pi_t} \frac{dF_t}{F_t} \right]$$

where the term on the right hand side is the covariance between the equity return and the stochastic discount factor. In particular:

$$\begin{aligned}
rp_t^D &= -\sigma_{\pi,t} \sigma'_{F,t} - \lambda_t \mathbb{E}_\zeta \left[-Z_t^{-\gamma} + 1 + Z_t^{\varphi-\gamma} - Z_t^\varphi \right] \\
&= -\sigma_{\pi,t} \sigma'_{F,t} + \lambda_t \mathbb{E}_\zeta \left[(Z_t^{-\gamma} - 1) (1 - Z_t^\varphi) \right] \\
&= \varphi \gamma \sigma^2 - q \frac{H_\lambda(\lambda_t)}{H(\lambda_t)} G_\lambda(\lambda_t) \sigma_\lambda^2 \lambda_t + \lambda_t \mathbb{E}_\zeta \left[(Z_t^{-\gamma} - 1) (1 - Z_t^\varphi) \right].
\end{aligned}$$

The instantaneous equity premium relative to the government bill rate, $rp_{b,t}^D$, is equal to rp_t^D minus the default premium $r_t^b - r_t$:

$$\begin{aligned}
rp_{b,t}^D &= -\sigma_{\pi,t} \sigma'_{F,t} - \lambda_t \mathbb{E}_\zeta \left[-Z_t^{-\gamma} + 1 + Z_t^{\varphi-\gamma} - Z_t^\varphi + p \left[Z_t^{-\gamma} - 1 - Z_t^{1-\gamma} + Z_t \right] \right] \\
&= -\sigma_{\pi,t} \sigma'_{F,t} + \lambda_t \mathbb{E}_\zeta \left[(Z_t^{-\gamma} - 1) ((1-p)(1 - Z_t^\varphi) + p(Z_t - Z_t^\varphi)) \right] \\
&= \varphi \gamma \sigma^2 - q \frac{H_\lambda(\lambda_t)}{H(\lambda_t)} G_\lambda(\lambda_t) \sigma_\lambda^2 \lambda_t + \lambda_t \mathbb{E}_\zeta \left[(Z_t^{-\gamma} - 1) ((1-p)(1 - Z_t^\varphi) + p(Z_t - Z_t^\varphi)) \right].
\end{aligned}$$

References

- ABEL, A. B. (1999): “Risk premia and term premia in general equilibrium,” *Journal of Monetary Economics*, 43, 3–33.
- ALMEIDA, H. AND T. PHILIPPON (2007): “The Risk-Adjusted Cost of Financial Distress,” *Journal of Finance*, 62, 2557–2586.
- BANSAL, R. AND A. YARON (2004): “Risk for the Long Run: A Potential Resolution of Asset Pricing Puzzles,” *Journal of Finance*, 59, 1481–1509.
- BARRO, R. J. (2006): “Rare disasters and asset markets in the twentieth century,” *Quarterly Journal of Economics*, 823–866.
- BARRO, R. J. AND J. F. URSÚA (2008): “Macroeconomic Crises Since 1870,” *Brookings Papers on Economic Activity*, 255–335.
- BEELER, J. AND J. Y. CAMPBELL (2012): “The long-run risks model and aggregate asset prices: An empirical assessment,” *Critical Finance Review*, 1, 141–182.
- BENZONI, L., P. COLLIN-DUFRESNE, AND R. S. GOLDSTEIN (2011): “Explaining asset pricing puzzles associated with the 1987 market crash,” *Journal of Financial Economics*, 101, 552–573.
- BOLLERSLEV, T., G. TAUCHEN, AND H. ZHOU (2009): “Expected Stock Returns and Variance Risk Premia,” *Review of Financial Studies*, 22, 4463–4492.
- CALDARA, D., J. FERNÁNDEZ-VILLAYERDE, J. RUBIO-RAMÍREZ, AND W. YAO (2012): “Computing DSGE models with recursive preferences and stochastic volatility,” *Review of Economic Dynamics*, 15, 188–206.
- CAMPBELL, J. Y. (2003): “Consumption-based asset pricing,” in *Handbook of the Economics of Finance*, ed. by G. Constantinides, M. Harris, and R. M. Stulz, Elsevier, vol. 1 of *Handbook of the Economics of Finance*, chap. 13, 803–887.
- CAMPBELL, J. Y., G. CHACKO, J. RODRIGUEZ, AND L. M. VICEIRA (2004): “Strategic asset allocation in a continuous-time VAR model,” *Journal of Economic Dynamics and Control*, 28, 2195–2214.
- CAMPBELL, J. Y. AND R. J. SHILLER (1988): “The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors,” *Journal of Financial Studies*, 1, 195–228.
- (1991): “Yield spreads and interest rate movements: A bird’s eye view,” *Review of Economic Studies*, 58, 495–514.
- CAMPBELL, J. Y. AND L. M. VICEIRA (2002): *Strategic asset allocation*, Oxford Univ. Press.
- CHACKO, G. AND L. M. VICEIRA (2005): “Dynamic Consumption and Portfolio Choice with Stochastic Volatility in Incomplete Markets,” *Review of Financial Studies*, 18, 1369–1402.

- COCHRANE, J. H. (1994): “Permanent and Transitory Components of GNP and Stock Prices,” *Quarterly Journal of Economics*, 109, 241–265.
- (2008): “The Dog That Did Not Bark: A Defense of Return Predictability,” *Review of Financial Studies*, 21, 1533–1575.
- DANIEL, K. AND S. TITMAN (1997): “Evidence on the Characteristics of Cross Sectional Variation in Stock Returns,” *Journal of Finance*, 52, 1–33.
- DUFFIE, D. AND L. G. EPSTEIN (1992a): “Asset Pricing with Stochastic Differential Utility,” *Review of Financial Studies*, 5, 411–436.
- (1992b): “Stochastic Differential Utility,” *Econometrica*, 60, 353–394.
- DUFFIE, D. AND C. SKIADAS (1994): “Continuous-time security pricing: A utility gradient approach,” *Journal of Mathematical Economics*, 23, 107–131.
- EPSTEIN, L. G. AND S. E. ZIN (1989): “Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework,” *Econometrica*, 57, 937–69.
- FAMA, E. F. AND R. R. BLISS (1987): “The information in long-maturity forward rates,” *American Economic Review*, 77, 680–92.
- FAMA, E. F. AND K. R. FRENCH (1988): “Dividend yields and expected stock returns,” *Journal of Financial Economics*, 22, 3–25.
- GABAIX, X. (2012): “Variable rare disasters: An exactly solved framework for ten puzzles in Macro-Finance,” *Quarterly Journal of Economics*, 127, 645–700.
- GLASSERMAN, P. (2003): *Monte Carlo Methods in Financial Engineering (Stochastic Modelling and Applied Probability) (v. 53)*, Springer, 1 ed.
- GOURIO, F. (2008): “Time-series predictability in the disaster model,” *Finance Research Letters*, 5, 191–203.
- (2012): “Disaster Risk and Business Cycles,” *American Economic Review*, 102, 2734–66.
- GUVENEN, F. (2006): “Reconciling conflicting evidence on the elasticity of intertemporal substitution: A macroeconomic perspective,” *Journal of Monetary Economics*, 53, 1451–1472.
- HALL, R. E. (1988): “Intertemporal Substitution in Consumption,” *Journal of Political Economy*, 96, 339–57.
- HANSON, F. B. (2007): *Applied Stochastic Process and Control for Jump-Diffusions: Modeling, Analysis, and Computation*, Siam.
- HAVRANEK, T., R. HORVATH, Z. IRSOVA, AND M. RUSNAK (2014): “Cross-Country Heterogeneity in Intertemporal Substitution,” Working Papers 2014/06, Czech National Bank, Research Department.

- JACKWERTH, J. C. AND M. RUBINSTEIN (1996): “Recovering Probability Distributions from Option Prices,” *Journal of Finance*, 51, 1611–32.
- JUDD, K. L. (1992): “Projection methods for solving aggregate growth models,” *Journal of Economic Theory*, 58, 410–452.
- (1998): *Numerical Methods in Economics*, MIT Press Books, The MIT Press.
- KEHOE, T. J. AND E. C. PRESCOTT (2002): “Great Depressions of the Twentieth Century,” *Review of Economic Dynamics*, 5, 1–18.
- KRAFT, H., T. SEIFERLING, AND F. T. SEIFRIED (2014): “Asset Pricing and Consumption-Portfolio Choice with Recursive Utility and Unspanned Risk,” SAFE Working Paper No. 52.
- KREPS, D. M. AND E. L. PORTEUS (1978): “Temporal Resolution of Uncertainty and Dynamic Choice Theory,” *Econometrica*, 46, 185–200.
- LINTNER, J. (1965): “The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets,” *Review of Economics and Statistics*, 47, 13–37.
- LUCAS, R. E. (1978): “Asset prices in an exchange economy,” *Econometrica*, 46, 1429–1445.
- MACAULAY, F. R. (1938): *Some theoretical problems suggested by the movements of interest rates, bond yields and stock prices in the United States since 1856*, NBER Books, National Bureau of Economic Research, Inc.
- MEHRA, R. AND E. C. PRESCOTT (1985): “The equity premium: A Puzzle,” *Journal of Monetary Economics*, 15, 145–161.
- (1988): “The risk equity premium: A Solution?” *Journal of Monetary Economics*, 15, 133–136.
- MIRANDA, M. J. AND P. L. FACKLER (2002): *Applied Computational Economics and Finance*, The MIT Press.
- MOSSIN, J. (1966): “Equilibrium in a Capital Asset Market,” *Econometrica*, 34, 768–783.
- NOWOTNY, M. C. (2011): “Disaster Begets Crisis: The Role of Contagion in Financial Markets,” .
- POHL, W., K. SCHMEDDERS, AND O. WILMS (2014): “Solving Asset Pricing Models with Recursive Preferences,” .
- RIETZ, T. A. (1988): “The equity risk premium: A solution,” *Journal of Monetary Economics*, 22, 117–131.
- SHARPE, W. F. (1964): “Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk,” *Journal of Finance*, 19, 425–442.

- SHILLER, R. J. (1981): “Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividends?” *American Economic Review*, 71, 421–36.
- TALLARINI, T. (2000): “Risk-Sensitive Real Business Cycles,” *Journal of Monetary Economics*, 45, 507–532.
- VAN BINSBERGEN, J. H., J. FERNÁNDEZ-VILLAVERDE, R. S. J. KOIJEN, AND J. F. RUBIO-RAMÍREZ (2012): “The Term Structure of Interest Rates in a DSGE Model with Recursive Preferences,” *Journal of Monetary Economics*, 59, 634–648.
- VISSING-JØRGENSEN, A. (2002): “Limited asset market participation and the elasticity of intertemporal substitution,” *J. Polit. Economy*, 110, 825–853.
- WACHTER, J. (2013): “Can time-varying risk of rare disasters explain aggregate stock market volatility?” *Journal of Finance*, 68, 987–1035.
- WACHTER, J. A. (2005): “Solving models with external habit,” *Finance Research Letters*, 2, 210–226.
- WEIL, P. (1989): “The equity premium puzzle and the risk-free rate puzzle,” *Journal of Monetary Economics*, 24, 401–421.

Table 1. Algorithm: Solving the time-varying disaster risk model

Initialization.

1. Define the approximation space $\lambda \in [\lambda_{min}, \lambda_{max}]$ and choose $n+1$ Chebyshev approximation nodes, $\lambda = \{\lambda_k : 0 \leq k \leq n\} \subset \lambda$. In particular we set $\lambda \in [0, m\sigma_\infty]$ where $\sigma_\infty = \sigma_\lambda \left[\frac{\bar{\lambda}}{2\kappa} \right]^{\frac{1}{2}}$ is the limiting standard deviation of the λ -process and $m \in \mathbb{N}^+$.
2. Choose functional forms for $\hat{H}(\lambda; \omega^H)$ and $\hat{G}(\lambda; \omega^G)$. In particular:

$$\hat{H}(\lambda; \omega^H) = \sum_{k=0}^n w_k^H \phi_k(\lambda), \quad \hat{G}(\lambda; \omega^G) = \sum_{k=0}^n w_k^G \phi_k(\lambda)$$

where ω^H and ω^G are $(n+1) \times 1$ vectors of unknown coefficients, and $\phi_k(\lambda)$ is the k -th member of the family of Chebyshev orthogonal polynomials.

Step 1. Use the reduced Hamilton-Jacobi-Bellman Equation (2.8) together with the approximant $\hat{H}(\lambda; \omega^H)$ and for $\lambda_0, \dots, \lambda_n$ compute the residual function:

$$\begin{aligned} \mathcal{R}^H(\lambda_k, \omega^H) = & \frac{f(C, V)}{V} \hat{H}(\lambda_k; \omega^H) + \left[(1-\gamma)\mu - \frac{1}{2}\gamma(1-\gamma)\sigma^2 + \lambda_k \mathbb{E}_\zeta(Z^{1-\gamma} - 1) \right] \hat{H}(\lambda_k; \omega^H) \\ & + \kappa(\bar{\lambda} - \lambda_k) \hat{H}_\lambda(\lambda_k; \omega^H) + \frac{1}{2}\sigma_\lambda^2 \lambda \hat{H}_{\lambda\lambda}(\lambda_k; \omega^H). \end{aligned}$$

Find the vector of unknown coefficients ω^H that make $\mathcal{R}^H(\lambda_k; \omega^H)$ “as close to zero as possible” by solving the resulting $(n+1)$ square system of non-linear equations.

Step 2. Take $\hat{H}(\lambda; \omega^H)$ from step 1. Compute the model’s risk-free rate using Equations (2.14) and (2.16). Use the approximant $\hat{G}(\lambda; \omega^G)$ and for $\lambda_0, \dots, \lambda_n$ compute the residual function for Equation (2.27) in Proposition (2.4):

$$\begin{aligned} \mathcal{R}^G(\lambda_k; \omega^G) = & \frac{1}{\exp(\hat{G}(\lambda_k; \omega^G))} - \hat{r}^f(\lambda_k, \omega^H) + \lambda_k \mathbb{E}_\zeta(Z^{(\varphi-\gamma)} - Z^{-\gamma}) + \varphi\mu + \frac{1}{2}\sigma^2\varphi(\varphi-1-2\gamma) \\ & + \left(\kappa(\bar{\lambda} - \lambda_k) + q \left(\frac{\hat{H}_\lambda(\lambda_k; \omega^H)}{\hat{H}(\lambda_k; \omega^H)} \right) \sigma_\lambda^2 \lambda_t \right) \hat{G}_\lambda(\lambda_k; \omega^G) + \frac{1}{2}\sigma_\lambda^2 \lambda_k \hat{G}_{\lambda\lambda}(\lambda_k; \omega^G) + \frac{1}{2}\sigma_\lambda^2 \lambda_k \left(\hat{G}_\lambda(\lambda_k; \omega^G) \right)^2. \end{aligned}$$

Find the vector of unknown coefficients ω^G that make $\mathcal{R}^G(\lambda_k; \omega^G)$ “as close to zero as possible” by solving the resulting $(n+1)$ square system of non-linear equations.

Evaluation. Choose a set of equally spaced evaluation nodes $\lambda^e = \{\lambda_k^e : 0 \leq k \leq n^e\} \subset \lambda$ with $n^e \gg n$ and compute approximation errors in both residual functions. If the errors do not fulfill a predefined error bound, start over at the initialization and change the degree of approximation, n .

Table 2. Parameters for the time-varying disaster risk model

The table replicates the parameter values used in Wachter (2013). In the model, time is measured in years and parameter values should be interpreted accordingly. The endowment of aggregate consumption is given by:

$$dC_t = \mu C_{t-} dt + \sigma C_{t-} dB_t - (1 - Z_t) C_{t-} dN_t$$

where N_t is a Poisson process with intensity rate λ_t . The process for the disaster intensity is given by:

$$d\lambda_t = \kappa (\bar{\lambda} - \lambda_t) dt + \sigma_\lambda \sqrt{\lambda_t} dB_{\lambda,t}.$$

The aggregate dividends in the economy D_t are equal to C_t^φ . The representative agent has recursive preferences defined by $V_t = \mathbb{E}_t \left[\int_t^\infty f(C_s, V_s) ds \right]$ with normalized aggregator:

$$f(C_t, V_t) = \frac{1 - \gamma}{1 - \psi^{-1}} \beta V_t \left[\left(\frac{C_t}{((1 - \gamma) V_t)^{\frac{1}{1 - \gamma}}} \right)^{1 - \frac{1}{\psi}} - 1 \right]$$

when $\psi \neq 1$ and

$$f(C_t, V_t) = \beta (1 - \gamma) V_t \left(\log C_t - \frac{1}{1 - \gamma} \log ((1 - \gamma) V_t) \right)$$

when $\psi = 1$.

Relative risk aversion, γ	3.0
Elasticity of intertemporal substitution, ψ	{0.5, 1, 1.5}
Rate of time preference, β	0.012
Average growth in consumption (normal times), μ	0.0252
Volatility of consumption growth (normal times), σ	0.02
Leverage, φ	2.6
Average probability of a rare disaster, $\bar{\lambda}$	0.0355
Mean reversion, κ	0.08
Volatility parameter, σ_λ	0.067
Probability of default, p	0.4

**Parameter values are in annual terms*

Table 3. Empirical and simulated moments for consumption and dividend growth

The table reports moments inclusive of jumps and conditional on no jumps from the simulated model, as well as empirical moments for the U.S. economy between 1947 and 2008, and between 1947 and 2013. The model is simulated at a monthly frequency and then aggregated to an annual frequency. All averages and volatilities are expressed in percentage terms. The simulation is carried out by first discretizing the continuous-time stochastic processes using a Euler scheme. For the model, the first column reports moments from simulated data that contains realizations of disasters. The second column reports moments of simulated data over years where no disasters occurred. $\rho_1(\cdot)$ denotes the coefficient of autocorrelation of order one. Consumption growth is simulated from:

$$d \log C_t = \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dB_t + \log(Z_t) dN_t,$$

while dividend growth is given by:

$$d \log D_t = \left(\mu_{D,t} - \frac{1}{2} \sigma^2 \varphi^2 \right) dt + \sigma \varphi dB_t + \varphi \log(Z_t) dN_t.$$

The last two columns report data moments calculated using annual observations constructed from monthly U.S. data from 1947 through 2008 and 2013 respectively.

Moment	Model		US Data	
	Population	Conditional	1947-2008	1947-2013
$\mathbb{E}(\Delta c)$	1.60	2.49	2.00	1.87
$\sigma(\Delta c)$	6.35	2.00	1.25	1.32
$\rho_1(\Delta c)$	-0.00	-0.01	0.35	0.46
$\mathbb{E}(\Delta d)$	4.17	6.47	2.49	2.40
$\sigma(\Delta d)$	16.52	5.20	5.26	6.19
$\rho_1(\Delta d)$	-0.00	-0.01	0.46	0.34

Table 4. Empirical and simulated moments for asset returns when $\psi = 1$

The table reports moments inclusive of jumps and conditional on no jumps from the simulated model when $\psi = 1$. The model is solved and then simulated at a monthly frequency. Simulated data are aggregated to an annual frequency. All averages and volatilities of asset returns are expressed in percentage terms. The second column reports moments from simulated data that contains realizations of disasters. The third column reports moments of simulated data over years where no disasters occurred. r^b denotes the gross return on government liabilities; r^e , the gross return on equities; r^f , the gross return on a risk free asset; and $(f - d)$ the log price-dividend ratio. $\rho_1(\cdot)$ denotes the coefficient of autocorrelation of order one. The last two columns report data moments calculated using annual observations constructed from monthly U.S. data from 1947 through 2008 and 2013 respectively.

Moment	Model		US Data	
	Population	Conditional	1947-2008	1947-2013
$\mathbb{E}(r^e)$	7.74	9.34	6.47	7.04
$\sigma(r^e)$	14.77	11.54	16.95	16.64
$\rho_1(r^e)$	-0.02	0.00	-0.00	-0.04
$\mathbb{E}(r^f)$	-0.21	-0.12	0.46	0.33
$\sigma(r^f)$	3.33	3.25	0.91	0.92
$\rho_1(r^f)$	0.95	0.94	0.43	0.46
$\mathbb{E}(r^b)$	0.99	1.36	1.37	1.12
$\sigma(r^b)$	3.79	2.00	2.88	2.91
$\rho_1(r^b)$	0.35	0.94	0.55	0.57
$\mathbb{E}(r^e - r^b)$	6.75	7.97	5.11	5.92
$\mathbb{E}(f - d)$	4.01	4.02	3.40	3.43
$\sigma(f - d)$	0.38	0.37	0.38	0.38
$\rho_1(f - d)$	0.92	0.92	0.89	0.88
Sharpe Ratio	0.48	0.66	0.30	0.36

Table 5. Empirical and simulated moments for asset returns for $\psi \in \{0.5, 1.0, 1.5\}$

The table reports moments conditional on no jumps from the simulated model. The model is solved for different values of the intertemporal elasticity of substitution. The model is then simulated at a monthly frequency and simulated data are aggregated to an annual frequency. All averages and volatilities of asset returns are expressed in percentage terms. r^b denotes the gross return on government liabilities; r^e , the gross return on equities; r^f , the gross return on a risk free asset; and $(f - d)$ the log price-dividend ratio. $\rho_1(\cdot)$ denotes the coefficient of autocorrelation of order one. The last two columns report data moments calculated using annual observations constructed from monthly U.S. data from 1947 through 2008 and 2013 respectively.

Moment	Model	Model	Model	US Data	
	($\psi = 0.5$)	($\psi = 1.0$)	($\psi = 1.5$)	1947-2008	1947-2013
$\mathbb{E}(r^e)$	9.57	9.34	9.15	6.47	7.04
$\sigma(r^e)$	4.41	11.54	13.55	16.95	16.64
$\rho_1(r^e)$	0.12	0.00	-0.01	-0.00	-0.04
$\mathbb{E}(r^f)$	1.73	-0.12	-0.74	0.46	0.33
$\sigma(r^f)$	3.99	3.25	3.04	0.91	0.92
$\rho_1(r^f)$	0.94	0.94	0.94	0.43	0.46
$\mathbb{E}(r^b)$	3.23	1.36	0.74	1.37	1.12
$\sigma(r^b)$	2.75	2.00	1.79	2.88	2.91
$\rho_1(r^b)$	0.94	0.94	0.94	0.55	0.57
$\mathbb{E}(r^e - r^b)$	6.34	7.97	8.41	5.11	5.92
$\mathbb{E}(f - d)$	3.67	4.02	4.25	3.40	3.43
$\sigma(f - d)$	0.20	0.37	0.41	0.38	0.38
$\rho_1(f - d)$	0.92	0.92	0.92	0.89	0.88
Sharpe Ratio	1.04	0.66	0.60	0.30	0.36

Table 6. Long-horizon excess return predictability

Excess returns measured over horizons of 1, 3 and 5 years are regressed on the log price-dividend ratio for simulated data from the model and from the sample data under different assumptions of the IES. We report estimated slope coefficient ($\hat{\beta}_1$), t-statistic, and R^2 statistic from a Newey-West estimator of the regression:

$$\sum_{i=1}^h (r_{t+i}^e - r_{t+i}^b) = \beta_0 + \beta_1 (f_t - d_t) + \epsilon_t$$

where r_t^e and r_t^b are the log returns on stocks and government bill respectively. Panel A reports results from simulated time series inclusive of jumps. Panel B reports results from simulated time series conditional on no disasters occurring. Panel C reports estimates using U.S. data for the period 1947-2008.

	Horizon in years								
	1			3			5		
	$\psi = 0.5$	$\psi = 1.0$	$\psi = 1.5$	$\psi = 0.5$	$\psi = 1.0$	$\psi = 1.5$	$\psi = 0.5$	$\psi = 1.0$	$\psi = 1.5$
Panel A: Model- Inclusive of jumps									
$\hat{\beta}_1$	-0.20	-0.15	-0.14	-0.55	-0.40	-0.38	-0.85	-0.62	-0.58
t	-69.83	-62.75	-59.41	-80.38	-77.60	-73.18	-79.44	-84.40	-79.96
R^2	0.18	0.15	0.13	0.37	0.36	0.33	0.46	0.49	0.46
Panel B: Model- Conditional on no jumps									
$\hat{\beta}_1$	-0.23	-0.16	-0.15	-0.64	-0.45	-0.42	-0.99	-0.70	-0.64
t	-210.52	-96.14	-84.58	-217.54	-133.10	-114.88	-163.70	-150.67	-132.04
R^2	0.59	0.25	0.20	0.76	0.53	0.46	0.75	0.66	0.60
Panel C: Data									
$\hat{\beta}_1$	-0.13			-0.29			-0.44		
t	-2.60			-2.92			-3.05		
R^2	0.10			0.20			0.26		

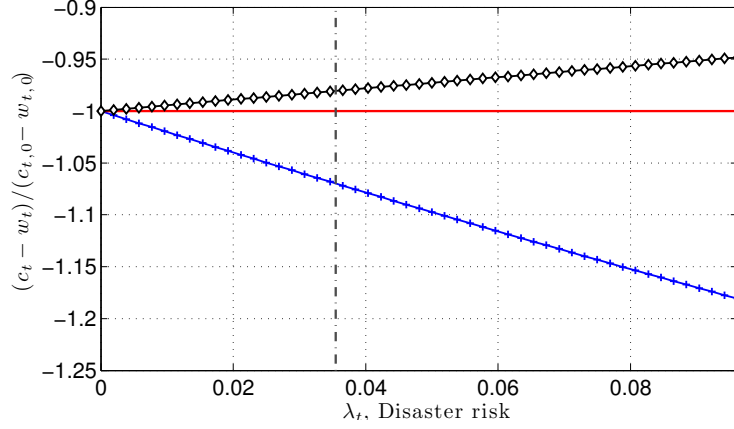
Table 7. Long-horizon consumption growth predictability

Aggregate consumption growth measured over horizons of 1, 3 and 5 years are regressed on the log price-dividend ratio for simulated data from the model and from the sample data under different assumptions of the IES. We report estimated slope coefficient ($\hat{\beta}_1$), t-statistic, and R^2 statistic from a Newey-West estimator of the regression:

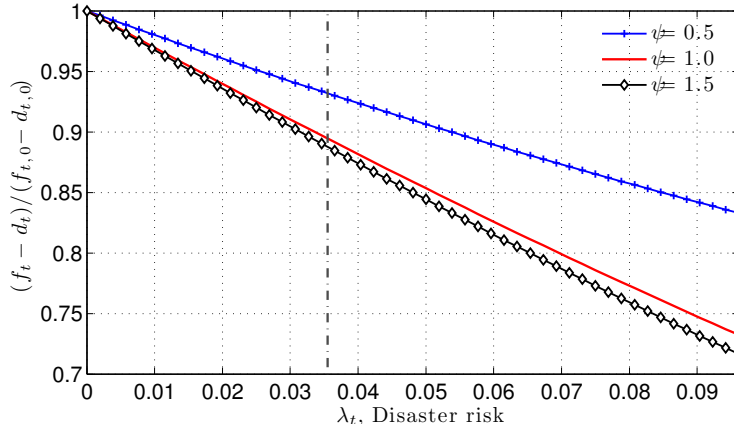
$$\sum_{i=1}^h (\Delta c_{t+i}) = \beta_0 + \beta_1 (f_t - d_t) + \epsilon_t$$

where Δc_t denotes the change in log consumption. Panel A reports results from simulated time series inclusive of jumps. Panel B reports results from simulated time series conditional on no disasters occurring. Panel C reports estimates using U.S. data for the period 1947-2008.

	Horizon in years								
	1			3			5		
	$\psi = 0.5$	$\psi = 1.0$	$\psi = 1.5$	$\psi = 0.5$	$\psi = 1.0$	$\psi = 1.5$	$\psi = 0.5$	$\psi = 1.0$	$\psi = 1.5$
Panel A: Model- Inclusive of jumps									
$\hat{\beta}_1$	0.036	0.020	0.017	0.099	0.054	0.048	0.152	0.083	0.073
t	18.778	18.615	18.648	20.461	20.300	20.332	20.062	19.907	19.933
R^2	0.014	0.014	0.014	0.034	0.034	0.034	0.048	0.048	0.048
Panel B: Model- Conditional on no jumps									
$\hat{\beta}_1$	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.002	-0.001	-0.001
t	-0.284	-0.313	-0.307	-0.935	-0.977	-0.970	-0.989	-1.034	-1.028
R^2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Panel C: Data									
$\hat{\beta}_1$	0.003			0.003			0.005		
t	0.807			0.287			0.372		
R^2	0.013			0.002			0.005		



(a) Log consumption-wealth ratio



(b) Log price-dividend ratio

Figure 1. Consumption-wealth ratio and price-dividend ratio. The graph shows normalized measures of the log consumption-wealth ratio (top panel) and the log price-dividend ratio (bottom panel) under different assumptions of the IES as a function of λ_t . The dotted vertical line is located at the unconditional mean of the disaster risk, $\bar{\lambda}$. For the collocation method an 11-degree approximation is used. For the discretization 11 nodes are used with $m = 11$. The evaluation step uses $n^e = 200$. The parameters for the consumption process, dividend process, disaster risk process and preferences are $\mu = 0.0252$, $\sigma = 0.02$, $\varphi = 2.6$, $\kappa = 0.08$, $\bar{\lambda} = 0.0355$, $\sigma_\lambda = 0.067$, $\gamma = 3.0$ and $\beta = 0.012$.

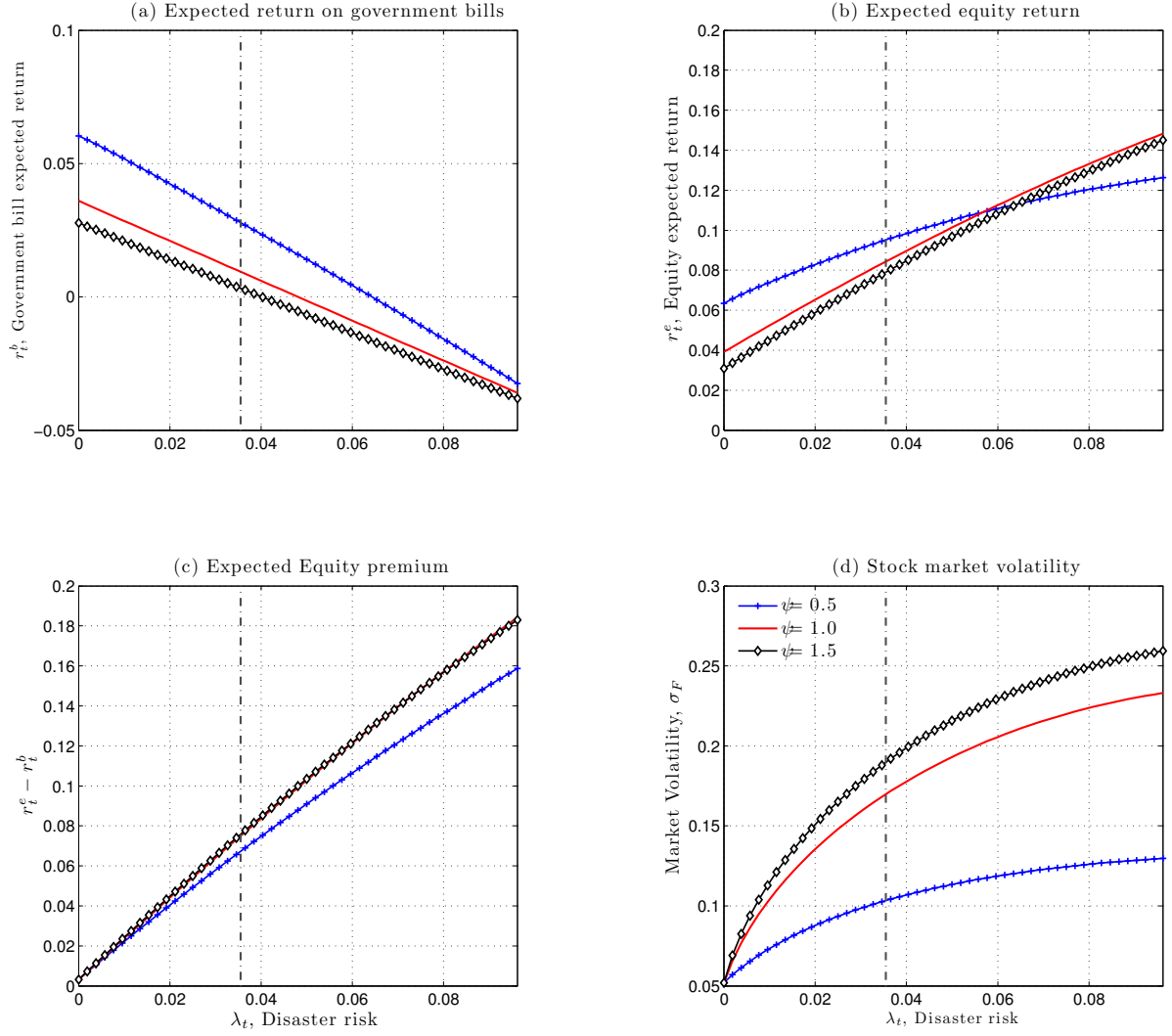


Figure 2. Asset prices, equity premium and aggregate volatility. The graph shows the expected return on government liabilities (top left panel), the expected return on equities (top right panel), the equity risk premium (bottom left panel) and the aggregate stock market volatility (bottom right panel) under different assumptions of the IES as a function of λ_t . The dotted vertical line is located at the unconditional mean of the disaster risk, $\bar{\lambda}$. For the collocation method an 11-degree approximation is used. For the discretization 11 nodes are used with $m = 11$. The evaluation step uses $n^e = 200$. The parameters for the consumption process, dividend process, disaster risk process and preferences are $\mu = 0.0252$, $\sigma = 0.02$, $\varphi = 2.6$, $\kappa = 0.08$, $\bar{\lambda} = 0.0355$, $\sigma_\lambda = 0.067$, $\gamma = 3.0$ and $\beta = 0.012$. The probability of default is set to $p = 0.4$.